A QUANTITATIVE BEHAVIORAL MODEL OF DEVELOPMENTAL STAGE BASED UPON THE MODEL OF HIERARCHICAL COMPLEXITY

Michael Lamport Commons and Patrice Marie Miller
Harvard Medical School

Behavior analytic theories have focused on explaining the acquisition of relatively simple behavior (the behavior of nonhuman species, of infants, and of individuals who are mentally retarded or autistic) rather than complex behavior. For these reasons, such theories have tended to become marginalized as far as developmental psychology as a whole is concerned. Developmental psychology as a whole has been concerned with what develops and in what sequence. The major theory that dealt with the possible sequences in which behavior is acquired has been the mentalistic theory of Jean Piaget (e.g., Piaget, 1954; 1976). We propose here a quantitative behavior-analytic theory of development that deals both with the sequences of development and with why development takes place. The theory presented here is behavioral because it makes only behavioral assumptions and avoids mentalistic explanations. By rejecting mentalism and substituting task analyses, we show that more complex behaviors combine and sequence less complex behaviors. This fact of hierarchical organization may be used to define the nature of stage and stage transition. Commons (Commons, Trudeau, et. al 1998) constructed the model of hierarchical complexity of tasks and their corresponding stages of performance using basically just three main axioms. As a consequence, there is only one possible stage sequence with gaps between the stages. The gaps cannot be filled with intermediate behaviors. The benefits for the field of psychology of having an analytic measure of stage are discussed.

A theory of development must be able to account for two aspects of behavior: a) what behaviors develop and in what order and b) why development takes place. It must be able to account for simple as well as complex behaviors. Behavior analytic theories of development have concentrated on explaining how development takes place (e.g., Bijou & Baer, 1961; Baer & Rosales, 1994). Development has been explained primarily in terms of contingencies of reinforcement. Such accounts have argued that the sequences in which behaviors develop are environmentally determined. Any particular behavior is viewed as being "shapeable" given the proper contingencies. As a result, sequences have been largely seen as arbitrary and easily changed. Behavior analytic theories have been better at explaining relatively simple behavior (the behavior of nonhuman species, of infants, and of individuals who are mentally retarded or autistic) rather than complex behavior. For these reasons, such theories have tended to become marginalized as far as developmental psychology as a whole is concerned.

Developmental psychology as a whole has been concerned with what develops and in what sequence. The major theory that dealt with the possible sequences in which behavior is acquired has been the mentalistic theory of Jean Piaget (e.g., Piaget, 1954; 1976). Skinner (1953) criticizes these types of theories as follows: "any mental event which is unconscious is necessarily inferential, and the explanation that makes use of it is therefore not based upon independent observations of a valid cause" (p. 39). A behavioral explanation is based instead on the relationship among detectable events, as will be discussed below.

We propose here a quantitative behavior-analytic theory of development that deals both with the sequences of development and with why

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Address reprint requests to Michael L. Commons, Department of Psychiatry, Harvard Medical School, Massachusetts Mental Health Center, 74 Fenwood Road, Boston, MA 02115-6196.

development takes place. The theory presented here is behavioral because it makes only behavioral assumptions and avoids mentalistic explanations. The theory also uses principles derived from quantitative analysis of behavior (e.g., Commons & Nevin, 1981) in that the assumptions are explicit and the measures of performance are quantitatively describable; neither are they limited by the earlier forays into quantification such as those of Hull (1943; 1952) or Piaget (Inhelder & Piaget, 1958; Piaget, 1954; Piaget, 1976; Piaget & Inhelder, with Sinclair-de Zwart, 1973). In Hulls' case, not only were there a very large number of postulates, but so many variables needed to be introduced as part of the postulate system. There was an ad hoc modification of the postulates to fit the data/entry.jsp?xrefid=149211.

PERTURBATIONS

In order to build a quantitative behavioral developmental theory it is necessary to start off by giving some informal definitions of the basic units to be studied. This theory starts by introducing the notion of a perturbation.

Perturbations as defined by Commons (LaLlave & Commons, 1996) are changes or disturbances in the universe that may be directly observed or may not. From a traditional point of view, the background for any perturbation is noise–statistically random fluctuations in the current state of affairs. That noise consists of changes in the situation that do not appear to be systematic.

EVENTS

Scientific accounts of behavior are built out of both analytical and empirical accounts of *events*.

Commons (LaLlave & Commons, 1996) sees that one problem that continually arises is what perturbations to consider as existing, or in other words, what constitutes an event. There only seems to be one necessary restriction on saying that something exists. The restriction is rather weak compared to those required by operationalism but strong with respect to intuitionism and phenomenonology. With the quantitative-behavioral-developmental theory that follows, we have to consider events as the basis. This notion is less restrictive than behaviorists' notions of stimuli and responses and so allows the theory to consider events that may not be clearly stimuli or responses. On the other hand, we do not want to make the mistake of Piagetians that thoughts, "schema," and verbalizations that belong to mental structures are the only causes of actions.

How do We Know that Something is an Event?

Events are potentially detectable perturbations. Perturbations are classed as *events* when they achieve some potential to be observed, witnessed, and in some way distinguished from the remaining noise by two independent paths of detection. The term event is used here to include all such perturbations, both public and private. The notion of paths of detection is not deniable or reducible lest we get into an infinite regress. These paths do not require direct observation. Note also that more experiencers or more experiences do not count as more independent paths.

Potential events may be inferred as long as there are *two distinct paths* leading to that inference, such as the case with electrons. Electrons may be detected through a multitude of paths by which inferences as to the existence of an "electron event" can be made. One can measure the magnetic moment of a single electron moving along a path in a magnetic field, the electric charge in an electric field, or the ionizing potential in a liquid hydrogen bubble chamber. There are numerous other ways of detecting the electron.

The reason two paths are required for events is because one path alone could mean that the perturbation could serve as its own causal explanation of itself. Some perturbations are deemed as having the status of being only singly detectable by one path. For example, if someone reports that the president is talking to them, there is one path, their report. They do not have a radio, telephone or any other such device and the president is nowhere close by. One other path is necessary to confirm that the president is actually talking to them and they are not reporting a hallucination. Behaviors and causes detected from a personal experience alone have this character. Only single path events have the character of hallucinations. Robert Stickgold (personal communication, 1999) has shown that people think that of what they think, see, and dream as "real" while thinking, seeing and dreaming. The status of events and perturbations is even more complex when activity is not potentially observable, as with gyrations and perturbations of the soul or will. These perturbations may be studied in theological and theosophical terms (Lowenthal, 1989). The best we can do within science is to discuss the report of these perturbations as data to be explained or refer to these perturbations in metaphorical terms.

Behavioral constructs (such as stimuli, behaviors, or consequences) are events. In the case of a verbal report, an observer may hear it. A microphone and meter will show it. There is a difference between the appearance of a perceived event and the actual event. Perceptual activity can transform events.

Illusions refer to those instances where organisms respond to the appearance of stimuli in ways that distort the physical properties of the objects or events. Let us say one was looking at a color patch and the person said, "I see the color brown." But the color brown has no unique spectral existence. The report of brown arises from an infinite number of mixes of spectral colors. Acting as if brown is a color is a distortion that it is simply a consequence of our visual aparati. Yet, with the same perceptual apparatus, people correctly report all the spectral colors. We consider that the perception or sense of free will is also a result of perceptual activity that transforms external and internal events. When discriminations are easy to make, people report that they have a sense of will when making correct choices. When discriminations are hard to make, people report that they have no sense of will in making their choices.

PUBLIC EVENTS

Public events are such that can be observed by more than one person (Skinner, 1957). External stimuli and behaviors are events. The two paths can be seen as follows. In addition to the person who observed their own behavior and the stimuli surrounding it, others may detect stimuli and behaviors. The behaviors may include language and emotional behavior as well as other responses.

Table 1: Ways of Knowing

	Ways of Knowing	Example of Fields Utilizing These Ways of Knowing	Number of Paths of Detections of Perturbations
1	Analytic: Proved material always true no matter what "data" or "experience" shows	Mathematics, Logic, Parts of Philosophy	No paths of detections of perturbations
2	Phenomenological: Experienced material a property of organisms and sometimes organisms interacting with environments.	Religion, Law, Art, Literature, Dance and Music	One independent path of detection. This means that if one observes an action and hypothesizes a cause, such as free will, then the putative cause may represent one path of detection. Detecting the behavior, however, does not prove that the hypothetical "causal" event is an actual event. If only one path is available, that is, if only one effect can be detected–that is the experience (and its report), there is no way to determine the cause of that experience. The experience is sometimes erroneously said to "cause itself."
3	Empirical: Resultant material from investigations moves scientific towards the truth.	Science, History	Two independent paths. An event can be said to be real in a scientific sense if and only if it is detectable by two independent paths. An independent second path for detecting the hypothesized causal event must be found.

Internal behavior such as one's heart beat are also public. One uses a stethoscope as a transducer to make the heart beat audible, and a electrocardiogram to make its activity visible.

PRIVATE EVENTS

Some private perturbations may also be events according to the above criteria. These include internal stimuli, states, and behaviors. Skinner (1953) writes that, "Eventually a science of the nervous system based upon direct observation rather than inference will describe the neural states and events which immediately precede instances of behavior." Internal stimuli might include internal pain (toothache) or pleasure, brain activity associated with dreams, or internal events modifying awareness of external events. For example, one of these internal events might include internal emotional activity that enhances attention to possible sources of reinforcement on one hand or distracts from attending to present events on the other hand. From our perspective, these states might include feelings, and tendencies to respond, such as attitudes and preferences. Internal behaviors might include images, illusions, thoughts, reflections, fantasies, delusions, hallucinations and intentions to act. For example, awareness may be considered internal behavior that is a response to either internal or external events. Awareness is sometimes described as the focusing of attention, or remembering internal events. Reports of awareness can be referred to as attentional behavior. That which is reported may acquire a relatively distinct and clear meaning. Presently, we only have one path to detect these internal perturbations--the subject's report. Because varying things in the environment affect reports of a number of these internal perturbations, one might think a possible second path may be inferred. Therefore, the reports are events, not perturbations (Skinner, 1957). Some of these events are already being detected by electronic-physiological means. With the potential to be detected, directly or indirectly, electrically or chemically, such internal perturbations may be classed as events and behaviors.

Private Events or Perturbations?

Where does this leave the cognitivist constructs of "internal mental life" that stem from fields such as cognitive development or psychodynamic theories? From a behavioral-developmental perspective, Commons (1991) and Gewirtz (1991) prefer to use alternatives that are based on events. These researchers may then take subjects' reports of internal events as potentially conditioned behavior just like any other. For example, **attachment** to an object can be a coherent system of responses cued and maintained by the appearance and behavior of an object person.

Another example might be traditional notions of the self. From a behavioral-developmental perspective, the *self* is viewed as an abstraction comprised simply of representations. Furthermore, a definition of a coherent system of responses might include: pervasively imitated behavior, rule-governed behavior, behavior in response to verbal communication, elicited emotional behavior, observed public behavior and unobserved private behavior (Commons, 1991; Gewirtz, 1991).

THREE WAYS OF KNOWING ABOUT DEVELOPMENT

With the definitions of perturbations and events, it is possible to show what are the minimum conditions necessary for having a quantitative behavioral developmental theory. One needs to recognize the different ways in which we might know and understand development. The argument is very simple. There are three ways of knowing as shown in Table 1. Knowledge is treated in a much more complex manner in philosophy. Here, the number of paths needed for detecting a perturbation is associated with the field and methodology that claims knowledge.

There can be combinations of ways of knowing such as 1 and 3, which defines most of science. Problems arise with combinations of 2 with 1 (Folk Psychology of Aristotle), and 2 with 3 (current mixes of experimental and phenomenological accounts of free will such as Libet's, 1985). These may lead to various dangerous policies and practices. That does not mean that 2 is not prized for itself. It is.

THE DETECTION OF EVENTS BY ORGANISMS

To make sure that all the assumptions are stated even those that are not formal but represent parts of the gist of the argument, we discuss detection of events by organisms. What is it that characterizes differences in performance as organisms evolve on one hand, and develop on another? Organisms, including people, are sensitive to events in the environment. Some aspects of events and some relationships between events can predict future events. Researchers consider those events and relationships as signals. Sensitivity to specific signals changes with developmental stage at which the organism functions. Developmental stage is discussed further on.

areas of: political development (Sonnert & Commons, 1994), workplace culture (Commons, Krause, Fayer, & Meaney, 1993), workplace organization (Bowman, 1996), relationships between more and less powerful persons such as doctors and patients; (Commons & Rodriguez, 1990, 1993; Rodriguez, 1989), decisions by therapists to report a patient's prior crimes (Commons, Lee, Gutheil, Goldman, Rubin, & Appelbaum, 1995), Kohlberg's moral interviews (Armon & Dawson, 1997; Dawson, 2000), views of the "good life" (Danaher, 1993; Dawson, 2000; Lam, 1994), Commons's (1991) attachment sequence, and extensions and adaptations of traditional Inhelder and Piaget balance beam and pendulum tasks (Commons, Goodheart, & Bresette, 1995; Inhelder & Piaget, 1958), and Loevinger's Sentence Completion task (Cook-Greuter, 1990).

Table 2: Stimulus, response and performance dimensions of tasks.

Name of dimension	Dimension	Definition
Hierarchical complexity	Stimulus	The number of times task-related actions act upon the output of lower-complexity actions in a chain of actions
Horizontal complexity	Stimulus	Number of stimuli and corresponding actions
Level of support	Stimulus	Transfer of stimulus control (level of support).
Reflectivity	Response	Degree of reflectivity of actions (from no reflectivity to reflections on methods for judgments)
Implicit or Explicit control	Response	Form of control over the operant responses
Behavioral stage and transition step of performance	Performance	Sensitivity to relationships in a task of given hierarchical complexity. A Rasch scaled score may also be found.
Bias	Performance	Tendency to assert relationship occurs

TASKS

One major basis for this developmental theory is task analysis. The study of ideal tasks, including their instantiation in the real world, has been the basis of the branch of stimulus control called Psychophysics. *Tasks* are defined as sequences of contingencies, each presenting stimuli and requiring a behavior or a sequence of behaviors that must occur in some non-arbitrary fashion. *Properties of tasks* (usually the stimuli, or the relationship among stimuli and behaviors) are varied and responses to them measured and analyzed. In the present use of task analysis, the complexity of behaviors necessary to complete a task can be specified using the complexity definitions described next. One examines behavior with respect to the analytically known complexity of the task.

THE SEQUENCE OF DEVELOPMENT

Dimensions of Tasks

The notion of hierarchical complexity to be introduced is to replace current accounts of development that rely on mentalistic notions (e.g., cognitive stages or schemas). The suggested task analyses can be carried out for any content area for which task analyses can be constructed. Thus far, we and various colleagues have carried them out in the

In this theory, how well an individual performs a task is postulated to be controlled by: 1) seven dimensions of tasks; 2) aspects of the situations in which tasks are presented; and 3) the reinforcement history of the individual. As Table 2 shows, we characterize tasks in terms of five stimulus and response dimensions. We characterize two performance dimensions. The first part of the discussion focuses on the dimensions of tasks because it is these dimensions, and particularly the first one (hierarchical complexity) that determine the sequence in which development takes place. These sequences occur in this order no matter how the reinforcement contingencies may favor out-of-sequence acquisition. Due to considerations of space, only the first three dimensions, which are also the most important, will be discussed here.

HIERARCHICAL COMPLEXITY: COMBINATIONS OF LOWER ORDER ACTIONS

There are a number of different kinds of combinations of lower order actions that can be combined into new stage behaviors (Binder, 2000): iterations, mixtures, chains and new stage behavior. Iteration is doing the same action over and over. For example, adding 2 + 3 + 4 + 1 is an iteration of adding. Mixtures of actions could include doing a problem set containing simple addition and simple multiplication tasks. Chains include the ordering of subtask actions but have an arbitrary order to them. For example, people learn to wash the dishes and then take out the trash. But in reality, people could take out the trash and then do the dishes if they so wished, making the order reversed. The tasks can be done in any order, but people choose to do them in a certain fashion. Finally, new stage behavior includes behaviors that are not in an arbitrary order.

THE MODEL OF HIERARCHICAL COMPLEXITY

The Model describes a new dimension of complexity that is at right angles to the traditional Horizontal complexity. It shows that all tasks fit in some sequence of tasks, making it possible to determine what order of hierarchical complexity an ideal action would have to be to address that task. Although this model has been previously described (Commons et al., 1998) the axioms are quantified here in an accurate, detailed form.

Axioms on the Relative complexity of actions:

The heart of the whole theory is discussed next. All the important properties that give rise to a coherent theory of hierarchical complexity depend on the following three axioms. They are not standard in other areas of mathematics or in much of behavior analysis.

In this task-complexity theory, for a task to be more hierarchically complex than another, the new task must meet three requirements. The new task-required action must satisfy the following axioms:

(1) *Formation of actions from prerequisites* (Axiom 6): The more hierarchically complex task and its required action must be *defined* in terms of the less hierarchically complex tasks and their required task actions:

$E_h = \{E_{one}, E_{two}...\}$; E_{one}, E_{two} are tasks, h refers to an order of hierarchical complexity.

This axiom is seen in programmed instruction (Holland & Skinner, 1961), in their discussion of prerequisites, and in Precision Teaching in the discussion of combinations being built out of elements (e.g. Commons & Richards, 2002; Kubina & Morrison, 2000). It is also basic to Piaget (e.g. Inhelder & Piaget, 1958) and Piaget's intellectual decedents (e. g. Campbell, 1991; Campbell & Bickhard, 1986; Tomasello & Farrar, 1986).

(2) *Relational Composition* (Axiom 7): A task-required action must organize two or more distinct, earlier actions in the chain. (R. M. Dunn, personal communication, January 26, 1986.)

$E_h = R(E_{one}, E_{two}...)$, where R is an ordering relation on two or more tasks.

This axiom is from Piaget (e.g. Inhelder & Piaget, 1958) in every version of his stage theory.

(3) *Order of Definition* (Axiom 8): The order of the organizing action and what it acts upon in the chain is fixed. The ordering must be *nonarbitrary.* That is translated as follows:

$E_h = -(i)R_i(E_{one}, E_{two}...)$, where i is the index indicating which order is defined by relation R_i, "-(i) " means it is not the case that for every i. In other words, it is not the case that every ordering of task execution exists. This axiom was developed by Commons (Commons, Trudeau, et al., 1998). Please see appendix for a discussion of this axiom.

To expand a little on these statements, the first axiom states that the very definition of a task-required behavior with a higher complexity must depend on previously defined, task-required behavior of lower complexity. Second, the higher-complexity task-required actions must coordinate the less complex actions. To coordinate actions is to specify the way a set of actions fit together and interrelate. The coordination specifies the order of the less complex actions. Third, the coordination must not be arbitrary. Otherwise the coordination would be merely a chain of behaviors. The meaning of the more complex task must not be severely altered by any non-specified alteration in the coordination.

Table 3. A sequence of behaviors placed into different orders of hierarchical complexity

Order	Name of Order of Hierarchical Complexity	Example
0	Calculatory	Simple Machine Arithmetic on 0's and 1's
1	Sensory & Motor	Seeing circles, squares, etc. or touching them.
2	Circular Sensory-motor	Reaching and grasping a circle or square. * * * * * O O O O O **# # # # #** **# / " } Q**
3	Sensory-motor	A class of filled in squares may be made
4	Nominal	That class may be named, "Squares"
5	Sentential	The numbers, 1, 2, 3, 4, 5 may be said in order
6	Pre-operational	The objects in row 5 may be counted. The last count called 5, five, cinco, etc
7	Primary	There are behaviors that act on such classes that we call simple arithmetic operations 1 + 3 = 4 5 + 15 = 20 5(4) = 20 5(3) = 15 5(1) = 5
8	Concrete	There are behaviors that order the simple arithmetic behaviors when multiplying a sum by a number. Such distributive behaviors **require** the simple arithmetic behavior as a prerequisite, not just a precursor 5(1 +3) = 5(1) + 5(3) = 5 + 15 = 20
9	Abstract	All the forms of five in the five rows in the example are equivalent in value, x = 5. Forming class based on abstract feature
10	Formal	The general left hand distributive relation is x * (y + z) = (x * y) + (x * z)
11	Systematic	The right hand distribution law is not true for numbers but is true for proportions and sets. x + (y * z) = (x * y) + (x * z) x ⊔ (y ⊓ z) = (x ⊓ y) ⊔ (x ⊓ z)
12	Meta-systematic	The system of propositional logic and elementary set theory are isomorphic x & (y or z) = (x & y) or (x & z) Logic ⇔ x ⊓ (y ⊔ z) = (x ⊓ y) ⊔ (x ⊓ z) Sets T(False) ⇔ ϕ Empty set T(True) ⇔ Ω Universal set
13	Paradigmatic	Distributive Systems are part of the Mathematical Paradigm. Mathematics integrates algebra, set theory, elementary probability theory, analysis, and based upon such an integration generates measure theory, and the mathematics used in physics.
14	Cross-paradigmatic-matic	Mathematics and Physics are integrated in the Mathematical Physics of Quantum Mechanics (Einstein, 1950; Planck,1922; Bohr, 1934) including the Standard Theory of Particles (Gell-Mann, 1964, Weinberg, 1983, Salam, 1972) Special and General Relativity Theory (Einstein, 1950)

Symbols
& = and
⇔ = is equivalent to
⊓ = intersection (overlap, elements in common)
⊔ = union (total elements)
T = Transformation of
ϕ = Empty set (no elements)
Ω = Universal set (all the elements there can be)
(Ex)= There exists some element x
(x) = For all x
(Hx) = The action on element x
T =Transformation of

One can tell whether one task is more hierarchically complex than another if they belong to the same task sequence. ((Need further explanation here)). In a single task sequence:

(4) $E_x < E_y$ or $E_x > E_y$. x, y are indexes standing for any task i; and < means the order of hierarchical complexity of E_x less than the E_y; >

means the order of hierarchical complexity of E_x more than the E_y.

Each task E_x may be decomposed into its less hierarchically complex component tasks E_{x11}, E_{x12} . . . Then the less hierarchically complex tasks E_{x11}, E_{x12} can be further decomposed into even lower hierarchically complex tasks E_{x21}, E_{x22} . This process can be carried out until one gets down to a task with a single simple action. If one of the E_{xij},= E_y, we are done. If one does not find such a E_{xij},= E_y,, look for an E_{yij},= E_x,

The Order of Hierarchical Complexity or just order of tasks is determined by the number of non-repeating *recursions* that constitute it. *Recursion* refers to the process by which the output of the lower-order actions forms the input of the higher-order actions. This "nesting" of two or more lower-order tasks within higher-order tasks is called *concatenation*. Each new, task-required action in the hierarchy is one order more complex than the task-required actions upon which it is built (tasks are always more hierarchically complex than their subtasks).

(5) The order, *O*, of hierarchical complexity of task *T* is denoted *O*(T), and defined as follows:

For a simple task t_i, $O(t_i)$ is 1.

(b) Otherwise, $O(E) = O(E') + 1$, where $O(E') = \max(C(E_1, C(E_2,. . .C(E_n))$ for all E_i in E.

In other words, the order of the next higher order task is one order of hierarchical complexity more than the next lower order task out of which it is built. If task E is built out of tasks of different orders of hierarchical complexity, then E' has the maximum order of all the tasks within it. Consider the example of distributivity, 3 * (9 + 2) = (3 * 9) + (3 * 2) = 27 + 6 = 33 where the numbers come from counting objects. The maximum order of the subtasks would be assigned by looking at the “adding” and “multiplying” actions (order 7), not the “counting” action (order 6).

Through such task analysis, the hierarchical complexity of any task in a task sequence may be determined. The hierarchical complexity of a task therefore refers to the number of concatenation operations it contains. An order-three task has three concatenation operations. A task of order three operates on a task of order two and a task of order two operates on a task of order one (a simple task).

There are a number of other “house-keeping” axioms listed in the appendices.

AN EXAMPLE OF DIFFERENT ORDERS OF HIERARCHICAL COMPLEXITY

In order to illustrate what a difference in the order of hierarchical complexity would look like, we will describe two specific tasks at different orders of hierarchical complexity. These tasks represent two different orders because the second one takes the actions of the first one and organizes them in such a way that it is not reducible to the first one.

The first task involves the development of an aspect of number knowledge. In the second grade, a child may add together two numbers. Some second graders may also multiply two numbers. We label such actions simple arithmetic operations (see line 8 of Table 3 for an example). A somewhat older child may carry out a second task, that is to combine addition and multiplication by carrying out a distribution action:

$$5 \text{ x } (1 + 3) = (5 \text{ x } 1) + (5 \text{ x } 3) = 5 + 15 = 20.$$

This hierarchically more complex action coordinates the less complex actions of adding and multiplying by uniquely organizing their sequence. The distributive action is therefore one order more complex than the acts of adding and multiplying alone. This action is required in both long multiplication and long division. Table 3 in its entirety shows the analytic sequence of the development of distributivity. For this sequence there are 12 orders of hierarchical complexity; in some sequences an additional two, even more complex, orders are added on at the end. Each order of hierarchical complexity is labeled in terms of a number (1-14 in this case) and an order name (See Table 4).

The lowest orders are characteristic of infancy (or of nonhuman species). The highest orders describe the complexity of tasks that can generally only be solved well into adulthood; this differs from the theory, for example, of Jean Piaget who postulated that the highest order of reasoning is reached in adolescence. In some respects, the orders here resemble the levels proposed by Fischer (1980;

Table 4: Stages described in the Model of Hierarchical Complexity

	Order or Stage	What they do	How they do it	End result
0	calculatory	Exact–no generalization	Human made program manipulate 0, 1	None
1	sensory & motor	Discriminate in a rote fashion, stimuli generalization, move	Move limbs, lips, eyes, head View objects and movement	Discriminative and conditioned stimuli
2	circular sensory-motor	Form open-ended classes	Reach, touch, grab, shake objects, babble	Open ended classes, phonemes
3	sensory-motor	Form concepts	Respond to stimuli in a class successfully	Morphemes, concepts
4	nominal	Find relations among concepts Use names	Use names and other words as successful commands	Single words: ejaculatives & exclamations, verbs, nouns, number names, letter names
5	sentential	Imitate and acquire sequences Follows short sequential acts	Generalize match-dependent behavior Chain words	Pronouns: my, mine, I; yours, you; we, ours; they, them
6	preoperational	Make simple deductions Follows lists of sequential acts Tell stories	Count random events and objects Combine numbers and simple propositions	Connectives: as, when, then, why, before; products of simple operations
7	primary	Simple logical deduction and empirical rules involving time sequence Simple arithmetic	Adds, subtracts, multiples, divides, counts, proves, does series of tasks on own	Times, places, counts acts, actors, arithmetic outcome from calculation
8	concrete	Carry out full arithmetic, form cliques, plan deals	Does long division, follows complex social rules, takes and coordinates perspective of other and self	Interactions, social events, what happened among others, reasonable deals,
9	abstract	Discriminate variables such as Stereotypes; logical quantification; (none, some, all)	Form variables out of finite classes Make and quantify propositions	Variable time, place, act, actor, state, type; quantifiers (all, none, some); categorical assertions (e.g. "We all die ")
10	formal	Argue using empirical or logical evidence Logic is linear, 1 dimensional	Solve problems with one unknown using algebra, logic and empiricism	Relationships are formed out of variables; words: linear, logical, one dimensional, if then, thus, therefore, because; correct scientific solutions
11	systematic	Construct multivariate systems and matrices	Coordinates more than one variable as input Consider relationships in contexts	Events and concepts situated in a multivariate context; systems are formed out of relations; systems: legal, societal, corporate, economic, national
12	metasystematic	Construct multi-systems and metasystems out of disparate systems	Create supersystems out of systems Compare systems and perspectives Name properties of systems: e.g. homomorphic, isomorphic, complete, consistent, commensurable	Supersystems and metasystems are formed out of systems of relationships
13	Paradigmatic	Fit metasystems together to form new paradigms	Synthesize metasystems of	Paradigms are formed out of multiple metasystems
14	cross-paradigmatic	Fit paradigms together to form new fields	Form new fields by crossing paradigms	New fields are formed out of multiple paradigms

Fischer, Hand & Russell, 1984), as well as others (e.g., Case, 1985; Pascual-Leone, 1984). The major difference is that their sequences are primarily empirically based and only secondarily rely on task analyses whereas the current sequence can be derived solely through analyzing tasks.

Orders of Hierarchical Complexity are Represented by Natural Numbers

It can be shown that the orders of hierarchical complexity are based on a system of natural numbers. Because of the nature of natural numbers, it can be shown that the separation between a less hierarchically complex task and a more hierarchically complex task is quantal in nature. If the orders of hierarchical complexity form an interval scale made up of the natural numbers, then there will be equal spacing of stages of performance of task items. We have developed a theorem and a proof of the theorem to demonstrate this.

Theorem 5: Hierarchical complexity is a linear((, interval)) scale of natural numbers . The scale of order of hierarchical complexity maps onto the natural numbers and the admissible transformations are of the form $mx + b$, where m is a natural number and b is any natural number. So if x is a natural number, so is the transformed $mx + b$. Hierarchical complexity, therefore, forms a linear natural number scale. This can be formally stated as follows:

N_i are natural numbers, N_i ?**N**

They are linear:

$N_j = mN_i + b$, where m and b are natural numbers, m ?**N**, b ?**N**

Proof:

By axiom 7 (originally from Commons et al., 1998): Each higher-order-of-complexity action, E_x, coordinates at least 2 lower-stage actions. The orders of hierarchical complexity increase in number of actions by at least twofold for every recursion. The order, O, is greater or equal to 2^O, the O^{th} power of 2. Such powers have the linear property, $y = mx + b$. In this case $y = O_j$, $x = O_i$. Hence the orders of hierarchical complexity have the linear property.

By this proof, the orders of hierarchical complexity are the natural numbers, not the ordinals. This can also be seen from the definition of the order of hierarchical complexity. From Expression 5, each subsequent order of higher complexity is just one more than the previous order of hierarchical complexity.

$O(E) = O(E') + 1$

$O(E') = O(E'') + 1$

$O(E'') = O(E''') + 1$

This reduces to: Order of Hierarchical Complexity = $O(E^{n}{}') = 2^n$ where each concatenation at order O is of 2 actions from the previous order, and $E^{n}{}'$ is E with n primes '.

Then $\log_2 2^n = n$; n is the order of hierarchical complexity

n belongs to the natural numbers.

In sum, this theory of hierarchical complexity suggests that:

The orders of hierarchical complexity are scaled by the natural numbers

Orders of hierarchical complexity are therefore an interval scale.

Because of Numbers 1 and 2, a number of implications for understanding stages and stage sequence follow:

3. Groups of tasks at different orders of hierarchical complexity should cluster in well-defined and equally spaced groups in the appropriate analysis. The analysis we have used is a Rasch analysis, to be described below.

4. Stages of performance are equally spaced in difficulty because orders of hierarchical complexity of the tasks are equally spaced.

5. All stage transitions are therefore equally difficult, from # 4.

6. Quantal nature of task hierarchy means there can be no intermediate performances. A task either meets conditions (1), (2), and (3) or does not.

7. There cannot be any other stages other than the 14 we have proposed except for ones beyond 14. We may have an error in the lowest stages, however.

Measuring Hierarchical Complexity: In our quantitative behavioral analysis of development, one would like to empirically verify three things. First, the Model of Hierarchical Complexity (MHC) predicts that the empirically-scaled task order should match the analytically-predicted sequence. Second, the MHC suggests that scaled values of the difficulty of the tasks of the same type and content should be some simple unidimensional transformation of linear. Third, the MHC predicts that the ordinal nature of hierarchical complexity should produce gaps in task difficulty. The most powerful quantitative analytic techniques that we have found for testing these predications are Rasch Analysis (1980) and the related Saltus analysis (Draney, 1996; Mislevy & Wilson, 1996; Wilson, 1989).

Rasch Analysis: Once a hierarchical order of tasks has been analytically determined, each participant is asked to solve all the tasks including the "easiest" and the "hardest." Participant responses are classified as either "right" (that is, fulfilling that task's contingencies) or "wrong" (failing to fulfill that task's contingencies). A Rasch (1980) analysis determines the probability of each participant performing a given task in terms of **task item difficulty** (delta or d) and **participant proclivity to respond correctly** (beta or b). See Appendix 2 for the specific model.

A Rasch and a Saltus Analysis of Two Different Tasks: A Saltus is related to a Rasch Analysis but allows one to scale performance of non homogenous groups by adding an additional parameter for group. It thereby can deal with the gaps in performances found between stages. We tested the three predictions by constructing two task sequences (Inhelder & Piaget, 1958) adapted by Commons (Commons, Goodheart & Bresette, 1995). One is the balance beam task and the other is the laundry task (based on an isolation of variables problem called the pendulum problem).

Here, both were pen and pencil instruments, consisting of a series of multiple choice problems of increasing hierarchical complexity. The tasks form a series because every higher order task has the lower order task embedded within it (see Siegler, 1986 for a review of various pre_formal and formal_balance beam tasks). Both tasks contained, at a minimum, items at the concrete, abstract, formal and systematic orders (or, as seen in Table 4, order #'s 8, 9, 10, and 11). Both adults and 5th- and 6th- grade children were participants.

For both the balance beam and the laundry problems, Quest software (Adams & Khoo, 1993) generated a separate Rasch model. The results support our prediction from MHC that the Balance Beam Task Series and the Laundry Task Series each measure a single dimension of performance. The tasks that were posited to be less complex were easier for subjects (see Commons, in preparation, for more details). The tight linear relationship between difficulty and hierarchical complexity as predicted by MHC (predictions 1 and 2 above) is shown in Figure 1. Scaled item difficulty (called Threshold) is plotted in log coordinates on the y-axis and Order of Hierarchical Complexity is plotted on the x-axis. Hierarchical complexity is also a log scale because order, n, is taken from the coordination of 2^n actions. Hence one would expect a straight line, which is pretty much what is obtained. In other words, as the order of hierarchical complexity increases, so does the difficulty of the item. The regression equation for difficulty (threshold) versus hierarchical complexity for the balance beam data is $r(16) = .92439$, $F(1,16) = 93.96473$, $r^2 = .85450$, $p < .0000$. Findings from the analysis of the laundry data are very similar (figure not shown here). The regression equation for laundry difficulty (threshold) verus order of hierarchical complexity is $r(22) = .918$, $F(1,22) = 118.417$, $r^2 = .843$, $p < .0000$.

A related Saltus analysis successfully tested for gaps in item difficulty that should be produced by the ordinal nature of hierarchical complexity, a third prediction of the General Model of Hierarchical Complexity. In addition to the tasks being properly ordered, the analysis showed that individuals who perform at lower orders of complexity never or rarely perform at higher orders of complexity, although the opposite is not true (Dawson, Commons & Wilson, in preparation). This provides further confirmation for the hierarchical ordering of tasks.

HORIZONTAL COMPLEXITY

Whereas Dimension 1 (Hierarchical Complexity) is postulated to be the most important dimension, as far as explaining performance, and many of the other dimensions are to some extent dependent on it, other dimensions are important as well. Horizontal complexity is the classical kind often found in information- processing theory. If one has a yes-no question, the answer contains 1 bit of information by definition. There are two alternatives, so the number of bits, n equals 2n alternatives. Each additional yes-no question adds another bit. The amount of this type of information required by a problem is the horizontal complexity. All computer programs can be reduced to a flat organization that can be represented by such yes-no questions (Campbell & Bickhard, 1986). How many bits a person can handle (somewhere between 5 and 9) seems to define the size of what is called short term memory. If the choices can be organized into larger classes (chunking) the amount of information that can be handled can increase.

Table 5 Levels of Support

Support number and Name	Change in measured complexity	Form of support	Action	Description
0.Manipulation	_3	Being moved though each step.	Literally being moved through each step of how to solve a problem.	Part of the stimulus is the push that guides the movement.
1. Transfer of stimulus control	_2	Being told each step (direct instruction).	Do a task based on a set of verbal instructions or other direct stimuli telling one what to do.	Train a discrimination with one set of stimuli on one task. Use the same set of stimuli to control performance in another task. Slowly remove first set of stimuli. This is like an errorless learning procedure (Moore & Goldiamond, 1964; Terrace, 1963).
2.Pervasive imitation	_1	Being shown.	Includes delayed imitation or observational learning (Gewirtz, 1969). The imitated action may be written, depicted or otherwise reproduced.	Fischer and Lazerson (1984) call this form of control the optimal level.
3. Direct	0	No help or support is given.	Problem-solving or hacking (without support).	Fischer and Lazerson (1984) call this the functional level. Most of Piaget's work was done at this level.
4.Problem finding	1	In addition, to not getting help, one must discover a task to answer a known question.	Persons are given an issue and they are asked to give a example of a problem that reflects that issue.	Arlin (1975, 1977, 1984) introduced postformal complexity (systematic order) by requiring the construction of a formal-operational problem without aid or definition.
5.Question finding	2	In addition, to not getting help and having to discover, one must discover the question	With a known phenomenon, people find a problem and an instance in which to solve that problem.	One has to discriminate the phenomenon clearly enough to create and solve a problem based on that discrimination.
6.Phenomenon finding	3	No direct stimulus control is possible without a description of phenomenon.	Discovering a new phenomenon.	No reinforcement history with phenomenon.

A good deal of variability in performance on tasks is due to variations in horizontal complexity. For example, one task may be 1 + 3 = ?. A more horizontally complex task might be 5 + 1 + 3 + 2 + 7 + 18 + 56 = ?. However, differences in horizontal complexity are not responsible for changes in hierarchical complexity. The two types of complexity are incommensurate and independent.

LEVEL OF SUPPORT

Dimension 3, or level of support, represents the degree of independence of the performing person's behavior from control by stimuli provided by others in the situation. There are 5 levels, and each level changes the relative difficulty of a task.. These levels are derived from Arlin (1975, 1984), Fischer, Hand and Russell (1984), Gewirtz (1969), and Vygotsky (1981a; 1981b). Table 5 lists the name, type of support at each level, and how each level of support changes the measured complexity relative to unaided problem solving. Then the action with respect to the subject is stated and some further description is provided.

These differing levels of support generate a partial model of how individuals' performances change as they begin to move from solving problems at a lower order of hierarchical complexity to solving problems at a higher order of hierarchical complexity. Specifically, when an individual is beginning to acquire behaviors that are appropriate for solving a problem at a higher order of hierarchical complexity, they may first require one or more levels of support. For example, it may be useful to see a worked example (1 level of support) for doing distribution as above before tackling 6 x (2 + 4) = ? the answer being:

(6 x 2) + (6 x 4) = 12 + 24 = 36.

Likewise, on a test, a problem may appear without support, examples or extra demands,

7 x (3 + 5) = ?

Last, for an extra credit project one might present a x (b + c) = ? This is one less level of support because participants have to generalize numbers to variables as in algebra.

Adjacent orders of hierarchical complexity cannot be split further, although if the actions organized were from two rather than one order lower, there would be intermediate organizing actions. What does occur is steps in transition between adjacent orders.

CONCLUSION

The theory presented here and in other papers on the Model of Hierarchical Complexity (Commons, et. al, 1998) makes six predictions, all of which Dawson, Commons and Wilson (in preparation) have confirmed:

There are exactly six stages in which we find participants performing, from the beginning of schooling to adulthood.

Sequentiality of stage is perfect.

Absolutely no mixing of stage scores for items takes place. A Saltus model (Wilson, 1989) shows that there is no continuity between the stage items.

Gaps in difficulty of items exist between stages. Not only is there no mixing but there are gaps.

These gaps are relatively equal, showing that the task demands of transitioning from one stage to another are similar regardless of the particular transition. These gaps have been shown using a Rasch analysis with a Saltus model.

People generally perform in a consistent manner across items from the same tasks of the same complexity. Most performances are predominantly at their most frequent stage of performance.

Behavioral approaches to development that go beyond Bijou and Baer (1961) are developing rather quickly. Some behavioral accounts have addressed development though adulthood for a broad range of people. We have based our account of development on five quantitative "laws" and have referred to a number of others. Before 1970, none of these laws had been formulated. Almost 30 years later, very few have been incorporated into behavioral accounts of development. The theory presented here has been expanded and deepened to account for much traditional developmental data while remaining entirely behavioral.

We have shown how the Model of Hierarchical Complexity leads to a quantal notion of behavioral stage. Removing any axiom from the above model leaves orders of hierarchical complexity (and therefore stage of performance) undefined. Adding more axioms would either reduce the generality of hierarchical complexity unnecessarily, or make the axioms inconsistent. No claim is made as to the uniqueness of the axiom system.

All tasks have some complexity associated with them. Thus, all tasks have stages associated with them. Because different orders of complexity require such large jumps in performance (Fischer et al., 1984) even though development may be continuous (Acredolo, 1995, Brainerd, 1978), it may appear as jumps or gaps (Commons & Calnek, 1984) on stage measures (Dawson, Commons, & Wilson, in preparation; Baer & Rosales, 1994). Commons and Calnek argue that continuous development is seen if task performance is measured often and with multiple items. If task performance is measured over a long period and only one measurement task is used, task performance may seem to occur in jumps. Commons and Richards (2002) address in detail the nature of transition.

Establishing an analytic measure of stage has many benefits for psychology. First, by classifying task complexity analytically, such a model produces measures that are independent of observation, and of actual subject performances. This leads to a greater degree of accuracy and consistency in stage measurement. Second, because the model defines a single sequence underlying all domains of development, it sets forth the core requirements of stages in every domain (see Kohlberg & Armon, 1984). Although many stage researchers posit more core requirements for stage, none require fewer. The set given by this model may allow for a systematic presentation of the consensus of theorists as to these core requirements. Indeed, the presence of a definitional set of axioms even makes it possible to determine whether a particular developmental theory also qualifies as a stage theory. For, according to this model, any theory that fails to account for the hierarchical complexity of task in the definition of development stage will by definition fail to yield results that are accurate, or even significant and meaningful as to order of developmental complexity.

Some doubt may remain as to whether there exists only one stage sequence. For example, if there were more than one way to perform a task, would this lead to alternative orders (and hence disagreement) as to the proper stage of a task and the true stage sequence? The answer, to a certain extent, is "yes". There will inevitably be some argument over the validity of certain task analyses. The fact that the analysis can be done by no means implies that it will be obvious or easy in all cases. It is possible to define our stage sequence such that it is generated from the task analysis with the shortest possible task chains, however. This will eliminate some ambiguity.

Additionally, it may be asked whether it is possible to know from a single task tree that another tree will not differ, such that complexity two on one task tree falls between complexity two and complexity three on another. The model, however, defines tasks that have two concatenations as tasks of complexity two, regardless of how difficult they may be to perform. "Falling between complexities" is therefore not a possibility. The General Model of Hierarchical Complexity shows that, because a single measuring system represents hierarchical complexity, only one stage sequence underlies all domains of development. For more than one sequence, another measure of hierarchical complexity would have to exist, although this by no means implies the structure-of-the-whole notion of Piaget. Such an alternative measure has not been identified, however. Moreover, because every task is of a certain order of hierarchical complexity, all tasks have a stage of performance associated with the response that they require for optimal resolution. Stage of performance on any given task will correspond to the order of hierarchical complexity of the task itself.

This model therefore answers some of the most fundamental questions that are asked about stage theories. By theoretically presenting a method for the analysis of tasks, and deriving an actual task chain, the model demonstrates that such chains exist. It also shows that stage sequence is invariable across all domains, because domain has been removed from the construction of the task sequence, and so has no implications for task complexity. Consequently, task complexity remains unchanged regardless of how broadly or narrowly domains are defined. Finally, the model offers an analytic model of stage development, based upon a set of mathematically grounded axioms. The axiomatic nature of the model entails that stages exist as more than *ad hoc* descriptions of sequential changes in human behavior, and formalizes key notions implicit in most stage theories. As such it offers clarity and consistency to the field of stage theory, and to the study of human development in general. It also lays the basis for a new form of computational complexity compatible with neural networks.

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APPENDIX 1

Non arbitrariness and its alternatives

The non-arbitrary axiom has been the newest and least clear to others. It is this very act of alternating been between actions in and what seems to be at first an arbitrary fashion that is the hallmark of transition. So why is the goal to order the actions in a non-arbitrary fashion? The problem is that all other forms of organization do not produce next order actions.

The properties of organization are listed from most restrictive to least restrictive: Fixed, Unique, Not Random, Random, Non

Arbitrary, Arbitrary. Each one of these ordering fails except for non-arbitrary.

Both Fixed and Unique are too restrictive because they do not allow for random orders, which are necessary to generate all possible combinations of actions at the systematic operational order. For example, the notion of a random variable would not be possible. There are infinite numbers of orderings that would generate the next hierarchal order. For example:

$x^*(y + z) = xy + xz$; $(x^*(y + z))^*1 = xy + xz$, etc. showing concrete required actions.

Non-random is also problematic. There are many variables at the formal order and above that require a random organization of actions. For example, a random selection of two variables may be used in a joint probability distribution. The problem with Random ordering is it leaves out fixed or finite or countable infinite orderings. For example, ransom orders would not included the orders of hierarchical complexity which are countable infinite. The problem with Arbitrary is that it does not specify an ordering at all. Hence it would not be able to produce the orders of hierarchical complexity. Non Arbitrary is the least restrictive but seems sufficient. We have not found or been told of any counter examples.

APPENDIX 2:

The Rasch analysis then fits the data to the following logistic model:

$$\Pr(X_{ni} = 0,1 / b_n, d_i) = \frac{\exp(X_{ni}(b_n - d_i))}{1 + \exp(b_n - d_i)}$$

That is, e is raised to the index function. That total quantity is divided by 1 + e to the difference between the values of b and d in the index function. The index function $X_{ni} = 0, 1$; X_{ni} is either 0 or 1 for a given value of b_n or d_i. X is the response (right or wrong) given by the subject to a task or item. The value d is the task or item difficulty. The value b is the subject proclivity.

APPENDIX 3:

Summary of axioms and theorems (For a more complete set of theorems, see Commons et al., 1998).

Order axioms

Based on the preceding definitions, it is now possible to begin to define a set of formal standards that must be satisfied to establish a consistent concept of stage. Here we will briefly describe the axioms; a more extensive description may be found in the appendix. The notion of *entity* serves as a point of departure. An entity is a set (or equivalence class) of tasks having the same order of hierarchical complexity. Entities must satisfy these three requirements for forming a *sequence*:

Axiom 1: Entities are non-trivial: Every entity must contain at least one potentially detectable task (i.e., for any entity *X*, there exists some task *x*).

Axiom 2: Entities are connected: There is no logical indeterminancy. The order of any entity is equal to, greater than, or less than the order of any other entity, but not more than one of these relations holds for any two entities.

Axiom 3: Entities are transitive: If the order of any entity A is greater than the order of some entity B, and the order of B is greater than some entity C, then the order of A is greater than the order of C.

Entity sequence axioms:

Axiom 4: Inclusivity: Entity *n* contains entity *n-1* inclusively. Inclusivity means that the higher-order action can do all that the lower-order actions can do and more. .

Axiom 5: Discreteness: The immediate successor of the entity of order *n* is the entity of order *n+1.* The entities are discontinuous.

Theorems resulting from the axioms

A system of orders of hierarchically tasks exists in any case in which all of the above axioms are satisfied. A stage of performance system parallels such a system. The following theorems are proofs derived from these axioms, and are demonstrated only informally.

Existence of Orders of Hierarchical Complexity and Resulting Stages

Theorem 1: Orders of Hierarchical Complexity exist. That collections of actions can be sequenced into orders of hierarchical complexity rests upon Axiom 6, which defines what is meant by qualitative difference. This discreteness or "gap" axiom requires that there be no interpolated action between sets of new required acts and the sets of previous order acts. For example, someone has performed an action (distribution) required by distribution task at the concrete order.

Corollary 1: Stages exist. If orders of hierarchical complexity of tasks exist, then there are actions that perform those tasks. The discovery of one case in which the gap axiom and the other order of hierarchical complexity axioms are satisfied is sufficient to logically demonstrate the existence of stages and stage sequences.

Theorem 2: Postformal hierarchical tasks and performance exist. As hierarchical complexity increases, the nature of the gap between each order of complexity changes. The gap from the primary to the concrete order involves only the coordination of addition and multiplication to form distributivity. In later-order gaps, such as the one from the systematic to the metasystematic order, at the metasystematic order, one has to create an entire metalanguage and set of metarules in order to coordinate the operations of a previous systematic order (Commons & Richards, 1984; The systematic order is also called the consolidated formal operational stage; Kohlberg, 1990; Pascual-Leone, 1984).

Making a deduction within a formal operational system requires formal operations. Showing that something is true about a formal-operational system requires systematic operations. Showing that something is true about a systematic formal-operational system requires metasystematic operations.

Theorem 3: A linear order may exist only within a single domain, on single sequences of tasks.

Axioms 4 through 7 are not so restrictive as to allow for this lattice structure, but are restrictive enough to require linear sequences within a single task sequence.

This result can be stated as follows. When one sequence of task performances in time is projected onto another sequence of task performances, the combined sequences do not necessarily form a linear order. The task sequences may have to be from the same domain, and the same subdomain.

Theorem 4. There is only one sequence of orders of complexity in all domains. The order numbers describe the same complexity of task-required actions irrespective of domain. Thus one can map any developmental sequence onto any other. This result does not imply synchronous development. Whereas the stage numbers may be the same, the stages of performance may develop at different times.

From an analytic perspective, the task requirements are constant and unvarying for different individuals regardless of how the subject feels about the task. The order complexity of each task within a sequence of tasks can be directly compared to the order of complexity for another set of tasks. The non-order of complexity aspects of tasks only make it more difficult to apply axioms 8 though 10.

*Theorem 5. (*Discussed in main text)

Theorem 6. Measures of performance: Whereas the gaps between orders of the complexity of tasks are discrete, measurement is continuous. Each discrete performance on a given stage task (actual or inferred) either succeeds (1) or fails (0).