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Using Rasch Scaled Stage Scores to Validate Orders of Hierarchical Complexity of Balance Beam Task Sequences

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These studies examine the relationship between the analytic basis underlying the hierarchies produced by the Model of Hierarchical Complexity and the probabilistic Rasch scales that places both participants and problems along a single hierarchically ordered dimension. A Rasch analysis was performed on data from the balance-beam task series. This yielded scaled stage of performance for each of the items. The items formed a series of clusters along this same dimension, according to their order of hierarchical complexity. We sought to ascertain whether there was a significant relationship between the order of hierarchical complexity (a task property variable) of the tasks and the corresponding Rasch scaled difficulty of those same items (a performance variable). It was found that The Model of Hierarchical Complexity was highly accurate in predicting the Rasch Stage scores of the performed tasks, therefore providing an analytic and developmental basis for the Rasch scaled stages.

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Introduction

The Model of Hierarchical Complexity

The Model of Hierarchical Complexity (Commons and Miller, 1998; Commons and Pekker, in press; Commons and Richards, 1984; Commons, Trudeau, Stein, Richards, and Krause, 1998) equates stage of performance on a task to the order of the hierarchical complexity of the tasks that the performance successfully addresses. The **order of hierarchical complexity** is measured by the number of recursions that the coordinating actions must perform on a set of primary elements. **Recursion** refers to the process by which the output of the lower-order actions forms the input of the higher-order actions. This "nesting" of two or more lower-order tasks within higher-order tasks is called **concatenation**. Each new, task-required action in the hierarchy is one order more complex than the task-required actions upon which it is built.

Formally, for a task to be more hierarchically complex than another, the new task must meet three requirements: First, a more hierarchically complex task and its required action is **defined** in terms of two or more less hierarchically complex tasks and their required task actions. Second, the more hierarchically complex task organizes or coordinates two or more less complex actions; that is, the more complex action specifies the way in which the less complex actions combine. Third, the coordination of actions that occurs has to be non-arbitrary. It cannot be just any chain of actions. Each new, task-required action in the hierarchy is one order more complex than the task-required actions upon which it is built (Commons, et al., 1998).

This hierarchy has been shown to account for performance in a variety of different domains, including: Physics tasks (Inhelder and Piaget, 1958: balance beam and pendulum (Commons, Goodheart, and Bresette, 1995); Kohlberg's moral interviews (Armon and Dawson, 1997; Dawson, 2000); Views of the "good life" (Danaher, 1994; Dawson, 2000; Lam, 1995); Loevinger's Sentence Completion task (Cook-Greuter, 1990), workplace culture (Commons, Krause, Fayer, and Meaney, 1993); Workplace organization (Bowman, 1996a, 1996b); Political development (Sonnert and Commons, 1994); Therapists' Decisions to report patient's prior crimes (Commons et al., 1995); and Relationships between more and less powerful persons such as doctors and patients (Commons and Rodriguez, 1990, 1993). We were particularly interested in examining performance on the balance-beam task series. Note that such an analysis has been performed for the Laundry Causality Series that is similar to Inhelder and Piaget's (1958) pendulum problem, and for the Counselor-Patient Informed-Consent series. The results were very similar.

In order to examine the complexity of such tasks, it is useful to illustrate the mathematical complexities that define The Model of Hierarchical Complexity. For example, consider the action of distributivity. Distributivity is the property of addition and multiplication on the real numbers that ensures that $a \times (b + c) = (a \times b) + (a \times c)$. Of course distributivity also plays a fundamental role in more general contexts, such as the complex numbers and the definition of rings in modern algebra. The distributive law serves as a motivation for a newer form of complexity, called hierarchical complexity, which we aim to describe in this paper.

In particular, the distributive law suggests that the task of evaluating $a \times (b + c)$ is more complex than the task of evaluating $(a + b) + c$ or even the two-part task of first evaluating $a + b$ and then evaluating $c \times d$. The evaluation of $(a + b) + c$ is no more complex than addition, performed either as $(a + b) + c$ or $a + (b + c)$; the organization of the two actions of addition is arbitrary. Similarly, in the two-part task, evaluating $a + b$ and then $c \times d$ yields the same result as first evaluating $c \times d$ and then $a + b$. Both of these are chain actions. On the other hand, the evaluation of $a \times (b + c)$ requires a non-arbitrary organization of addition and multiplication, or, equivalently, the distributive law, and is therefore more complex than addition or multiplication. In modern algebra, the non-arbitrary coordination of addition and multiplication leads to the definition of rings, and the expressions in ring theory are usually more complex than the expressions in group theory (which involve only one operation).

We refer to addition and multiplication as actions, a term that is commonly used by developmental psychologists to refer to events that produce outcomes or, equivalently, accomplish certain tasks. The study of tasks appears in psychophysics (a branch of stimulus control theory in psychology) (Green and Swets, 1966; Luce, 1963) and in artificial intelligence (Goel and Chandrasekaran, 1992), and in general, actions may be attributed to organisms, computers, or others. Clearly actions may be combined to produce new, more complex actions (Binder, 2000). Our goal is to describe how to measure the complexity of an action and to relate it to the complexity of other actions.

In the literature, two types of complexity have been identified: horizontal (traditional) and vertical (hierarchical) (Commons et al., 1998). (For a review of these definitions, see, e.g., Wolfram, 2002, and Kauffman, 1993). Roughly speaking, in traditional complexity, the complexity of an action is determined by the number of times a specific subaction is repeated. In hierarchical complexity, the complexity of an action is determined by the non-arbitrary way in which the subactions are organized. In particular the order of hierarchical complexity of an action is one greater than the order of hierarchical complexity of its subactions, provided they are organized in a non-arbitrary way.

To illustrate one difference between traditional and hierarchical complexity, consider the action A of evaluating $1 + 2$ and the action B of evaluating $(1 + 2) + 3$. The traditional complexity of A is smaller than the traditional complexity of B since the action of addition is executed less often in A than in B; on the other hand, since A differs from B only in how many times addition is executed, but not in the organization of the addition, A and B have the same hierarchical complexity. This example shows that the two types of complexity are independent and incommensurate.

The axioms in the next three sections build on Piaget (e.g., Inhelder and Piaget, 1958) and Piaget's intellectual descendants (e.g., Campbell, 1991; Campbell and Bickhard, 1986; Tomasello and Farrar, 1986); they are also fundamental to stacked neural networks (Commons and White, 2003). Other applications include programmed instruction in the discussion of prerequisites (Holland and Skinner 1961) and precision teaching in the discussion of combinations being built out of elements (e.g., Commons and Richards, 2002; Kubina and Morrison, 2000). Although the model itself has been previously described (Commons et al., 1998), the formal, axiomatic version is presented here.

Actions

We begin by defining the fundamental terms. In a given system, there exist certain tasks that are to be accomplished. These tasks are accomplished via task-actions. Formally, *a task-action*, often abbreviated simply as an action, is defined inductively. There exists a unique *simple action* $\tilde{A}$, which is the simplest action possible in a system. This is in agreement with Luce's choice theory (Luce, 1959). Every other action A consists of at least two (and possibly infinitely many) previously defined actions and a *rule* for organizing those previously defined actions. Thus, every nonsimple action A is an ordered pair $A = (\{A_1,\ldots\}, R\}$ where the first component is a multi-set of at least two previously defined actions A_i composing A and R is the rule for organizing those actions.

There are two categories of rules: chain rules and coordination rules. In a nonsimple action $A = (\{A_1, \ldots\}, R)$, a *chain rule* R is simply a sequential execution of the actions A_i in some order, but the order of the executions *does not matter*. That is, regardless of the order in which the subactions are executed, the result of A is achieved. *A coordination rule*, on the other hand, requires the execution of the actions A_i in some specific, non-arbitrary order, so that the order *does matter*.

We now formalize these notions. Suppose first that A consists of finitely many subactions, i.e., $A = (\{A_1, A_2, \ldots, A_n\}, R)$. Given a permutation $\sigma = (i_1, i_2,\ldots, i_n)$ of the numbers $1, 2,\ldots, n$, the execution of the A_i according to σ is simply

$A_{i_1}, A_{i_2}, \ldots, A_{i_n}$.

In this notation, the rule *R* is a chain rule if the outcome of *A* is the same for all *n*! permutations of the numers 1, 2,…, *n*. That is, the outcome of the order of actions,

$A_{i_1}, A_{i_2}, \ldots, A_{i_n}$

is the same for all permutations $(i_1, i_2, \ldots, i_n)$ of 1,2,…, *n*. The rule *R* is a coordination rule if this is not the case; i.e., if there exists at least one permutation $\tau = (j_1, j_2, \ldots, j_n)$ of the numbers 1, 2,…, *n* so hat the execution of the actions A_i according to τ, i.e.,

$A_{j_1}, A_{j_2}, \ldots, A_{j_n}$,

is *not* the same as the outcome of the action *A*. Hence, the outcome of A_i is given by at least one, but *not all*, permutations of the A_i. We extend similarly to the cases where *A* consists of infinitely many actions.

We summarize these definitions as the first three action axioms; we will refine them in the following section.

(A1) There exists a simple action Ã.

(A2) Every action *A* is either simple (so *A* = Ã,) or composed of at least two previously defined actions $\{A_1,\ldots\}$ and a rule *R* for organizing those actions (so $A = (\{A_1,\ldots\}, R)$).

(A3) Each rule is either a chain or a coordination.

To motivate the definition of hierarchical complexity in the next section, we will rely on the following example.

Example 1. Let + and × denote the traditional addition and multiplication on the real numbers, and let ⊕ and ⊗ denote the traditional addition and multiplication of variables (having values, say, in the real numbers). Then, consider the following four actions.

(a) $A = (\{+, \times\}, R_A)$ consisting of 1 + 2 (i.e., adding the numbers 1 and 2) followed by 3 × 4 (i.e., multiplying the numbers 3 and 4). Clearly, the order in which the two subactions are executed does not matter: adding 1 and 2 and then multiplying 3 and 4 yields the same results, namely 3 and 12, as multiplying 3 and 4 and then adding 1 and 2. Thus, *A* is a chain action.

(b) $B = (\{+, \otimes\}, R_B)$ consisting of 1 + 2 followed by $x \otimes y$. Again, the order in which the two subactions are executed does not matter: adding 1 and 2 and then multiplying *x* and *y* yields the same results, namely 3 and *xy*, as multiplying *x* and *y* and then adding 1 and 2. Thus, *B* is also a chain action.

(c) $C = (\{+, \times\}, R_C)$ consisting of the expression 2 × (3 + 4). This is not a chain, for the order of the subactions matters: if we multiply 2 and 3 first and then add 4, we get 10, not 14, which is the answer dictated by rule R_C (i.e., adding 3 and 4 first and multiplying the result by 2). Thus *C* is a coordination, not a chain.

(d) $D = (\{\oplus, \otimes\}, R_D)$ consisting of the expression x ⊗ (1 ⊕ 2). Notice that since the expression involves real numbers *and* variables, we must necessarily use ⊕ and ⊗ and not simply + and ×. In particular, because the distributive law dictates that

$$x \otimes (1 \oplus 2) = (x \otimes 1) \oplus (x \otimes 2),$$

we cannot replace ⊕ by +. This observation will be important in the next section. As in the previous case, it is clear that *D* is a coordination action.

(e) $E = (\{\oplus, \otimes\}, R_E)$ consisting of the expression x ⊗ (y ⊕ z). This is exactly the same as (c) but at a more abstract level, and is, therefore, a coordination rule.

Hierarchical Complexity

To each action *A* we wish to associate a notion of that action's hierarchical complexity, *h*(*A*). Since actions are defined inductively, so is the function h, known as the order of the hierarchical complexity. For a simple action A, $h(A) = 0$. For a non-simple action, $A = (\{A_1,\ldots\}, R)$, we have to consider several cases. To get an intuitive idea, we analyze the complexity of the actions in Example 1.

Example 1 (Continued). Let m be the hierarchical complexity of + and ×, the traditional addition and multiplication on the real numbers, and let n be the hierarchical complexity of the operations $\oplus$ and $\otimes$, the traditional addition and multiplication of variables. Intuitively we understand that $m < n$.

(a) Action A is a chain. The order in which the sub-actions forming the chain are executed can be changed without impacting the product of the actions. Therefore, executing A does not require any skill beyond the execution of each of the subactions individually.

(b) Similarly, B is a chain rule, but executing B requires being able to multiply at the abstract level (which is more complex than adding at the primary level), and so $h(B) = \max((h(+), h(\otimes)) = h(\otimes) = n$. Notice that unlike action A, action B consists of subactions of different complexities.

(c) Observe now that action C coordinates two subactions of the same order, namely m. Since the order in which the two subactions are executed is nonarbitrary, the hierarchical complexity of this action is higher than the complexity of its subactions: $h(\mathrm{c}) > \max(h(+), h(\times)) = m$.

(d) As we remarked in Example 1, it may seem at first that action D coordinates two actions of different orders, + of lower order and $\otimes$ of higher order. However, due to the distributive law, it actually coordinates two actions of the same order, i.e., n. In particular, we observe that a coordinating action, at least in arithmetic, necessarily coordinates subactions of equal order. As in the previous case, we see that $h(D) > \max(h(\oplus), h(\otimes)) = n$.

(e) Lastly, as in (c), it is clear that $h(E) > \max(h(\oplus), h(\otimes)) = n$.

This analysis illustrates that the only way to raise hierarchical complexity is by coordinating actions of lower complexity. Moreover, coordination requires the subactions to be of equal orders. In light of Example 1, we now state the hierarchical complexity axioms which incorporate the action axioms (A1) - (A3).

Hierarchical Complexity Axioms

(HC1) There exists a simple action $\tilde{A}$, and $h(\tilde{A}) = 0$.

(HC2) Every nonsimple action $A = (\{A_1, \ldots\}, R)$ is either a chain of at least two previously defined actions of arbitrary orders of hierarchical complexity or a coordination of at least two previously defined actions *all of* which have the same order of hierarchical complexity.

(HC3) For a nonsimple action $A = (\{A_1, \ldots\}, R)$, $h(A) = \max_i h(A_i)$ if A is a chain, and $h(A) = h(A_1) + 1$ if A is a coordination.

Notice that by Axiom (HC2), a coordination action $A = (\{A_1, \ldots\}, R)$ necessarily coordinates subactions of equal orders of hierarchical complexity (i.e., $h(A_1) = h(A_2) = \ldots$), and so the order of hierarchical complexity of A is one higher than the order of hierarchical complexity of all its subactions. In particular, in the last equation in Axiom (HC3) we may replace A_1 by any subaction of A and still obtain the same result.

As a consequence of these axioms, we see that if we let $\mathbf{A}$ denote the collection of all actions in a given system, then the hierarchical complexity is a function $h: \mathbf{A} \to \mathbb{N}$, where $\mathbb{N} = \{0, 1,\ldots\}$ is the set of natural numbers (and zero) under the usual ordering. From the properties of the natural numbers, we immediately obtain the following four essential properties of hierarchical complexity.

Consequences of Hierarchical Complexity Axioms

(HC4) (Discreteness) The order of hierarchical complexity of any action is a nonnegative integer. In particular, there are gaps between orders.

(HC5) (Existence) If there exists an action of order n and an action of order $n + 2$, then there necessarily exists an action of order $n + 1$.

(HC6) (Comparison) For any two actions A and B, exactly one of the following holds: $h(A) > h(B)$, $h(A) = h(B)$, $h(A) < h(B)$. That is, the orders of hierarchical complexity of any two actions can be compared.

(HC7) (Transitivity) For any three actions A, B, and C, if $h(A) > h(B)$ and $h(B) > h(C)$, then $h(A) > h(C)$.

In light of Appendix A that describes the orders of hierarchical complexity for, among others, arithmetic tasks, we can assign the exact natural numbers corresponding to the orders of tasks in Example 1.

Example 1 (Continued). According to Appendix A, both + and × have order 7, i.e., primary, while ⊕ and ⊗ have order 9, i.e., abstract.

(a) Since A is a chain, $h(A) = \max(h(+), h(\times)) = 7$, i.e., also primary.

(b) Since B is a chain, $h(B) = \max((h(+), h(\otimes)) = 9$, i.e., also abstract.

(c) Since C is a coordination, $h(C) = \max(h(+), h(\times)) + 1 = 8$, i.e., concrete.

(d) Since D is a coordination, $h(D) = \max(h(\oplus), h(\otimes)) = 10$, i.e., formal.

(e) Again, since E is a coordination, $h(E) = \max(h(\oplus), h(\otimes)) + 1 = 10$, i.e., formal.

Stages

The notion of stages is fundamental in the description of human, organismic, and machine evolution. Previously it has been defined in some ad hoc ways; here we describe it formally in terms of the Model of Hierarchical Complexity. Given a collection of actions $\mathbf{A}$ and a participant S performing $\mathbf{A}$, the *stage of performance* of S on $\mathbf{A}$ is the highest order of the actions in $\mathbf{A}$ completed successfully, i.e., it is

$$\text{stage}(S, \mathbf{A}) = \max \{h(A) \mid A \in \mathbf{A} \text{ and } A \text{ completed successfully by } S\}.$$

Thus, the notion of stage is discontinuous, having the same gaps as the orders of hierarchical complexity. This is in agreement with previous definitions (Commons et al., 1998; Commons and Miller, 2001).

We will return to the notion of stage in the experimental results. Appendix A lists the stages described by the Model of Hierarchical Complexity.

Measure of Hierarchical Complexity

We define the *measure of complexity at order n*, denoted by φ_n, as the minimum number of simple actions required to complete an action of order n. By axioms (HC2) and (HC3), an action of order n organizes at least two actions of order $n - 1$, each of which in turn organizes at least two actions of order $n - 2$, and so forth, until we reach the lowest-order, simple actions. Consequently, given the inductive definition of the hierarchical complexity orders, it is not surprising that $\varphi_n = 2^n$. Formally, a zero-order action, consists of at least one simple action, so $\varphi_0 = 1 = 2^0$. For the inductive case, suppose $\varphi_{n-1} = 2^{n-1}$. Because by axioms (HC2) and (HC3), an action of order n is either a coordination of at least two actions of order $n - 1$ or a chain which includes an action of order n (and hence eventually is composed of at least two actions of order $n - 1$), we have $\varphi_n = 2\varphi_{n-1} = 2^n$, by induction.

Discerning the relationship between properties of stimulus inputs and their corresponding responses can provide us with a great deal of knowledge about how machines, animals and people work. In the field of psychophysics, for example, investigations into these types of interconnections has led to advances in our understanding of sensory, perceptual, and cognitive processes. Naturally, a more comprehensive understanding of the properties of inputs facilitates this fruitful research into the relationship between stimuli and responses. These studies examine how successfully The Model of Hierarchical Complexity characterizes the input of the complexity of mathematical problems. A successful model would explain the developmental trajectory of problem solving skills—at least for these kinds of tasks over a greater developmental range, in greater detail and with more accuracy than now exists. This information could possibly aid in future research on how individuals may acquire more advanced skills for accurately solving these problems.

If the Model of Hierarchical Complexity described here is accurate in assessing the hierarchical complexity of a task, then a task that it orders as highly complex should be more difficult to perform than a task that it orders as less complex. The tasks that are more complex and, therefore, difficult should generally be performed less successfully than those which are less complex, and therefore, easier among any population of individuals, given everything else equal. In this study, Rasch measurement models (Bond and Fox, 2001; Rasch, 1960), tested the efficacy of the Model of Hierarchical Complexity. The test examined whether those tasks that fewer people in the participant pool could perform were tasks of higher levels of complexity in the Model of Hierarchical Complexity. More specifically, using the Rasch model, the tasks administered were hierarchically ordered by how many participants successfully answered them. The Rasch model is especially proficient at making this determination because it transforms raw data into unidimensional, abstract, linear, equal-interval scales if the data fit the model. Equality of intervals is achieved through log transformations of raw data odds. The Rasch model is the only model that provides the necessary objectivity for the construction of a scale that is separable from the distribution of the attribute in the persons it measures. If the tasks measured on this Rasch scale correspond to the same way they were measured on the Model of Hierarchical Complexity, the Mode of Hierarchical Complexity would emerge as a highly effective way to assess the hierarchical complexity of a mathematical problem.

The Rasch Model

A Rasch model produces an objective, additive scale that is independent of the distributions of the particular items used and of the particular participants tested. It can be used to analyze a large variety of human sciences data. This model, through the use of probabilistic equations, converts raw ratings of items into scales that have equal intervals if the data fit the model. Such a scale can then be used as a type of objective ruler against which to measure the data on items as well as on respondents (Andrich, 1988). Statistically speaking, this scale will be linear (Wright and Stone, 1979). As a result, a change of difficulty of an item of 1 logit is the same going from –2 to –1 as going from 0 to +1.

After analyzing data with a Rasch model, a number of questions can be answered. First, where on the scale does each item fall? In this case, this may indicate the measured stage as defined above in the section labeled stage, rather than just the designated order of hierarchical complexity for each of the items. Second, what is the spacing of scaled values of the items of differing orders of complexity? Third, to what extent do the scaled stage values of these items fit on the same scale? The answer to these questions will yield a scale of stage of the items.

The Rasch Model and Hierarchical Complexity

The hierarchical complexity model makes four predictions that should be evident in real world data. First, in interviews that probe for stage of performance, the scoring of the stage derived from the Model of Hierarchical Complexity should provide the clearest and most reliable account among all scoring systems. Second, the empirically scaled orders of complexity of tasks should match the analytically predicted sequence of orders of complexity of these tasks. Third, the empirically scaled orders of complexity of tasks of the same type and content should be related by a simple unidimensional linear transformation. Fourth, the empirically scaled orders of tasks should produce gaps due to the natural number scale of hierarchical complexity. The first prediction has been verified in Dawson (2002), and so we focus on the last three.

We use Rasch analysis (Rasch, 1966; Rasch, 1960) to test these predictions. (The relationship between the Rasch model and conjoint measurement is discussed in Brogden, 1977; Fischer, 1968; Keats, 1967, 1971; for more on the Rasch model as an application of conjoint measurement to empirical data, see Young, 1972; Luce and Tukey, 1964; and Perline, Wright, and Wainer, 1979). Suppose we have a collection of tasks with hierarchical orders of complexity d_j ($1 \leq j \leq J$) and a collection of participants with proclivities to answer correctly b_i ($1 \leq I \leq J$); the parameters d_j and b_i are determined analytically. The Rasch model predicts that participant i completes task j correctly with probability

$$P\left(X_{ij}=1\right)=\frac{\exp\left(b_i-d_j\right)}{1+\exp\left(b_i-d_j\right)}.$$

Clearly, the probability that participant i fails to complete task j correctly is

$$P\left(X_{ij}=0\right)=1-P\left(X_{ij}=1\right)=\frac{1}{1+\exp\left(b_i-d_j\right)}.$$

Justification of Using a Rasch Model

The first assumption of the Rasch model deals with the unidimensional nature of the hierarchy of items as it has been generated using the Model of Hierarchical Complexity. In applying the model to development, it is assumed that each task of increasing hierarchical complexity should be more difficult. The assumptions of the Rasch model also include: (1) the performances are drawn from a single population with a common set of proficiencies and that tasks can be effectively mapped onto an interval measurement scale; (2) there are sufficiently dense distributions of person proficiencies and item difficulties with a correspondingly sufficient overlapping of error distributions around items (Pelton, personal communication, February 10, 2005; Pelton and Bunderson, 2003).[1]

Expanding on (2), it would be possible that the distance between two "adjacent" items is so great relative to the sample sizes and the error distributions, and that the tails would not overlap sufficiently to allow the Rasch model to accurately estimate relative positions of items or persons on the measurement scale. When this is the case the 'local' person response patterns (i.e., sub-patterns of responses within a hierarchically ordered response vector) will be Guttman-like (1950), then the second assumption is violated and the positioning of items and persons will be on an ordinal scale.

It is anticipated that participant data may violate the first assumption. If the participants perform at different stages, one might consider that they really belong to different groups, each group reflecting a given stage. Nevertheless, despite the possibility of these violations of the assumptions, and because of the expected measurement noise, Rasch analysis can be used to obtain useful evidence in support of hierarchical complexity (Bond and Fox, 2001) as discussed below. Because the orders of hierarchical complexity are ordinals and have gaps between them, we predict the Rasch model estimates may produce gaps in measured stage of performance. That is, we might find clusters of Rasch scores of tasks of approximately the same hierarchical complexity with a few Rasch scaled task scores between them.

If there was no probabilistic process (no error or measurement variance), then the data would have a Guttman pattern. From a Rasch perspective, this would mean that items at each order of hierarchical complexity would be at the same

1 Pelton (personal communication, February 18, 2005) and Linacre personal communication (February 19, 2005) say that the Rasch model scaled response error variances are modeled as heteroscedastic (unequal variances). For a typical dichotomous response of probability p of a correct answer, the binomial error variance is $p(1-p)$. Hence, there is the assumption that there is a binomial distribution of erroneous responses centered above each item that is placed on the true measurement scale and that these distributions are unequal (heteroscedastic).

point, with an infinite distance to the next level of difficulty. This is also true for person proclivity (stage) levels. If items perform exactly the same (again, meaning there is no error variance) when presented to a sample of persons, WINSTEPS (Linacre, 2004) would not be able to provide scale estimates—and would therefore, provide no evidence of even a local measurement scale. For Guttman scaling to apply, it is the case that the orders of hierarchical complexity are sufficiently distinct. Also, the person samples would need to be sufficiently dense around the items so that they will be placed appropriately on the scale in a hierarchical fashion. Arbitrary gaps (intervals) between clusters of Rasch scores would occur when there are Guttman-like response vector components for the persons who have achieved a developmental stage.

When this type of local discontinuity is expected to occur because of theorized developmental shifts, then the Rasch model can confirm such, and then subsequently be used to generate several independent measurement scales – one for each cluster of items within the developmental stages (using selected subsets of the data). Rasch scaling works for stage data because, even in Piagetian stage data, there is always some empirical departure from an exact stage structure, i.e., there is always some error variance. What could be viewed as a flaw in Piagetian theory is used as a basis for estimating relatively how far apart the stages are on the latent variable. Provided there is a useful level of construct relevant noise in the observations, Rasch scaling is possible.

Method

Participants

Convenience sampling was used to gather 121 predominantly Caucasian, middle class participants (80 female, 38 male, 3 did not report gender), whose ages ranged from 7 to 66 years ($M = 29.22$, $SD = 12.98$). Participants were given unlimited time to complete a multiple choice pen and paper instrument containing primary, concrete, abstract, formal, systematic and metasystematic order balance beam problems.

Instrument

The balance-beam task series is a pen and paper instrument that consists of a series of multiple-choice problems of increasing hierarchical complexity. The tasks form a series because every higher order task has the lower order task of the previous task embedded within it (see Siegler, 1986; for a review of various pre-formal and formal-balance beam tasks). In the series used, there were 5 primary order of hierarchical complexity items, 5 concrete order items, 5 abstract order items, 9 formal order items, 8 systematic order items, and 10 metasystematic order items. The instrument is also available in its entirety at: http://www.dareassociation.org/.

The following is an example of a concrete order of hierarchical complexity item. The concrete-order of hierarchical complexity balance beam task requires the coordination of two operations. First, the total weight on each side of the beam must be equal. On one side of the fulcrum, there is a single weight. This weight must be balanced by a stated weight plus some unknown additional weight. The participant determines the unknown additional weight by subtracting the stated amount of weight (3) from the total (7).

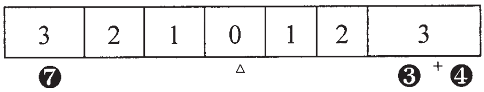
Example:

Choose the number that balances the balance beam:

3	2	1	0	1	2	3
❼			△			❸ + *x*

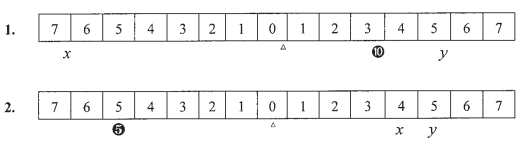
- The beam is represented by the horizontal table.
- The middle of the beam is at the 0 point, which is indicated by the △ symbol.
- The numbers to the left and right of the balance point (fulcrum, 0) on the beam are the distances.
- The numbers in the black circles below the beam are the added weights.
- The value x is the amount of weight needed to balance the beam. In the example, the amount needed is 4.

Solution:



The following is an example of a systematic order of hierarchical complexity item. The systematic Balance Beam task presents two formal order balance beams to the participant. The same two unknown quantities (weights or distances) must be determined for both beams. In order to find the unknowns, the participant is required to solve a set of simultaneous equations (or find the solution by trial and error from among a finite list of possibilities, a potential problem, discussed further below). The participant must relate two formal operational equations to solve these simultaneous equations.

Example: You are still on Counter-Earth, where negative numbers are used for counting. Your mission now is to figure out how to make these beams balance. Here are two sample balance beams. The arithmetic operation that defines torque is the same for each. The mathematical relationship that governs the balancing of each system is also the same for each.



Solution:

The unknown weight *x* is equal to:	The unknown weight *y* is equal to:
a. 5	a. 1

Method of Administration

The adult participants received the complete range of developmental tasks, from primary to metasystematic. The children received all developmental tasks except the metasystematic task, which we deemed to be too long and too difficult for them to perform. Half of the adults received the instrument in class from the experimenters and completed it at home. At a later date, these participants recruited the other half of the adult sample, becoming experimenters in their own right.

Results

The analysis was carried out in four steps. First, to find the scaled difficulty of the items (see Table 1), a Rasch model of the data was generated with WINSTEPS (Linacre, 2004). The consistency of the hypothesized difficulty order (by the Model of Hierarchical Complexity) of the items was tested with a Rasch analysis. The reliability of the item estimates from the Rasch analysis was .98 with mean infit mean-square of .59 (range .24 to 1.41). This supported Commons' claim that the balance-beam task series measures a single dimension of performance. Cronbach alpha was .93, indicating the items were effective is separating participants along the hierarchical complexity continuum.

Second, the Model of Hierarchical Complexity was highly accurate in predicting the stage of performance on tasks as shown in Figure 1. There was an extremely strong relationship between the order of hierarchical complexity (a task property variable on the x-axis) of the tasks and the corresponding Rasch scaled difficulty of those same items (a performance variable on the y-axis), $r(42) = .879$, $F(1, 40) = 136.22$, $p < .0005$, $r^2 = .773$ (see Figure 2). The mean scaled score was $M = .00$ ($SD = .41$), and the mean order of hierarchical complexity was $M = 9.9$ ($SD = 1.70$).

Third, a univariate analysis showed that the order of hierarchical complexity of the items made a large difference in stage spacing, $F(4, 30) = 261.09$, $p < .0005$. Since we were testing the null hypothesis that stage spacing was equal, the large effect size, $r^2 = .976$, suggests that stage spacing was not equal.

Fourth, part of the spread of Rasch scores within an order of hierarchical complexity was explained by the horizontal complexity of the items. Horizontal complexity is the sum of the number of actions at a given order of hierarchical complexity required to complete a task. In addition to the hierarchical complexity as a predictor of Rasch scaled scores of difficulty, a multiple regression

Table 1

Balance-beam Task Series fit statistics

Entry Number	Raw Score	Count	Measure	Error	Infit Mnsq	Ranks
4	107	121	.43	.08	.24	prim4
3	109	121	.42	.07	.24	prim3
1	113	121	.40	.07	.23	prim1
5	113	121	.40	.07	.25	prim5
2	114	121	.39	.07	.26	prim2
10	115	121	.39	.07	.23	concr5
12	116	121	.38	.07	.33	abstr2
7	119	121	.37	.07	.22	concr2
8	119	121	.37	.07	.21	concr3
6	120	121	.36	.07	.21	concr1
13	120	121	.36	.07	.28	abstr3
11	123	121	.35	.06	.25	abstr1
9	125	121	.34	.06	.54	concr4
20	135	121	.31	.06	.86	form2n1
15	139	121	.29	.06	.92	abstr5
14	143	121	.28	.05	.89	abstr4
22	154	121	.25	.05	.95	form2n3
21	169	121	.21	.05	.62	form2n2
19	174	121	.20	.05	1.18	form1n4
16	194	121	.16	.04	1.41	form1n1
17	201	121	.15	.04	1.30	form1n2
23	213	121	.13	.04	.81	form2n4
18	236	121	.09	.04	1.05	form1n3
24	237	121	.08	.04	.97	form2n5
26	270	121	.03	.04	.65	sys1q1y
25	277	121	.02	.04	.54	sys1q1x
28	312	121	–.03	.04	.75	sys1q2y
27	321	121	–.04	.04	.77	sys1q2x
29	337	121	–.06	.04	.97	sys3q1x
30	340	121	–.06	.04	1.11	sys3q1y
31	343	121	–.07	.04	.98	sys3q2x
32	355	121	–.08	.03	.98	sys3q2y
33	888	121	–.68	.05	.32	met1
40	888	121	–.68	.05	.35	met8
35	890	121	–.68	.05	.33	met3
39	890	121	–.68	.05	.34	met7
37	891	121	–.68	.05	.35	met5
38	891	121	–.68	.05	.35	met6
41	891	121	–.68	.05	.35	met9
34	892	121	–.68	.05	.35	met2
42	894	121	–.69	.05	.34	met10
36	899	121	–.70	.05	.50	met4
MEAN	357	121	.00	.05	.59	
SD	308	0	.41	.01	.35	

analysis revealed that the horizontal complexity of items within each order of hierarchical complexity for this task was also a predictor. Whereas the Order of Hierarchical Complexity had $\beta = 2.143$, $t = 11.508$, $p < .0005$; the Horizontal Complexity also made a significant but smaller contribution, $\beta = 0.253$, $t = 2.079$, $p = 0.044$. With both factors in the multiple regression, r increased from .879 to $r(41) = .880$, $F(2, 37) = 66.804$, $p < .0005$, $r^2 = .774$. The item with the lowest order of horizontal complexity and of the lowest order of hierarchical complexity was expected to have the lowest Rasch scaled score, followed by the item with the next highest horizontal complexity of that order,

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Note: Items are abbreviated as Primary Order 7 = pri7, Concrete Order 8 = con8, Abstract Order 9 = abs9, Formal Order 10 = for10, Systematic Order 11 = sys11, Metasystematic Order 12 = met12. Each # on the left of the *y* axis represents two participants and a single dot represents one participant.

Figure 1. Variable map for order of hierarchical complexity

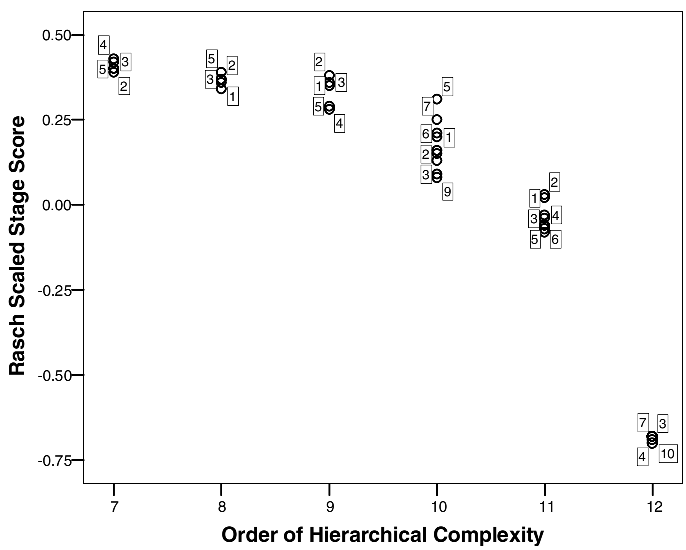
and so on. The multiple regression revealed that the Rasch analysis was highly accurate in ordering these items according to the two degrees of complexity (see Figure 2).

The generality of these results is due to the fact that the resulting scaling of item difficulty obtained with a Rasch analysis is independent of the distribution of the particular participants used when the data fit the model (Bond and Fox, 2001). The Rasch model enables us to estimate subject ability and item difficulty independently of each other (Andrich, 1988). Among the participants there needs to be a range of performance so that some fail almost all tasks and others get almost all tasks correct. In this task series, it was clear that the items were rather easy for participants, as indicated by the distribution of participant scaled scores (see left hand side of Figure 1). Several participants were scored at or around the systematic items, thereby indicating that they would answer items of systematic order complexity correctly about 50% of the time.

Summary and Predictions of Model of Hierarchical Complexity

In sum, the theory of hierarchical complexity predictions were supported:

1. The orders of hierarchical complexity were shown to be scaled by the natural numbers. Because of this, a number of implications for understanding stages and stage sequence follow:
2. Sequentiality of stage of items should be near perfect as has been shown here and elsewhere (e.g., Dawson, Commons, Wilson and Fischer, 2005).
3. Because orders of hierarchical complexity of the tasks are ordinals, groups of tasks at different orders of hierarchical complexity should cluster in well-defined groups. Using Rasch analysis, this trend was found in the study reported here and elsewhere (e.g., Dawson, Commons, Wilson, and Fischer, 2005). The trend of like-ordered items clustering together is apparent in Figure 1.



Note: Rasch Scaled Stage Score as a Function of Order of Hierarchical Complexity, $r(42) = .879$, $F(1,40) = 136.22$, $p < .0005$, $r^2 = .773$. For the Horizontal Complexity, $\beta = 0.253$, $t = 2.079$, $p = 0.044$.

Figure 2. Balance-beam Task Series Regression. The numbers show the effects of horizontal complexity, with the lower numbers representing less horizontally complex problems.

4. Quantal nature of task hierarchy means there can be no intermediate single performances. A task either meets conditions (1), and (2), or does not. But performances may be intermediate because they can be mixtures of stages. We found intermediate performances
5. People may perform in a consistent manner across items from the same tasks of the same complexity. Most Rasch scaled performance scores align with their most frequent stage of performance.

The Model of Hierarchical Complexity predicted that the Rasch model-generated order scores would correspond to a hierarchically ordered task sequence. Order of hierarchical complexity of tasks strongly predicted corresponding Rasch scaled scores, with approximately 88% of the variance in Rasch scale scores accounted for by the order of hierarchical complexity. This suggests that, within a domain and on a given series of tasks, order of hierarchical complexity accounts for most of the variance in performance. Rasch analysis was used here to relate performance on a task series to hierarchical complexity of the problems within the series. This demonstrates the feasibility of: a) constructing series of mathematical problems in an area of the researchers' interest; b) predetermining the hierarchical complexity of the problems in that series; c) and measuring performance on those problems using Rasch models. The findings of this study and the establishment of the Model of Hierarchical Complexity as accurate in assessing the input of task complexity could thus aid our analysis and understanding of test performance.

Acknowledgments

The idea for using task sequences was suggested by Kurt W. Fischer (Fischer and Bidell, 1998; Fischer and Bullock, 1981; Fischer, Knight and van Parys, 1993). The task series was created by Commons and Goodheart and administered by Commons. Portions of this paper were presented by Commons, Richards, Trudeau, Dawson, and Goodheart (1997) and the same authors at The Association for Behavior Analysis in May 1994 in Atlanta.

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Appendix A

Behaviors may form classes. Stimuli may be placed into classes both functionally and analytically.

Order of Hierarchical Complexity	Name	Example
0	Calculatory	Simple Machine Arithmetic on 0's and 1's
1	Sensory or Motor	Either seeing circles, squares, etc. or instead, touching them. ○ ■
2	Circular Sensory- Motor	Reaching and grasping a circle or square. ○ ■
3	Sensory-Motor	A class of filled in squares may be formed ■ ■ ■ ■ ■
4	Nominal	That class may be named, "Squares"
5	Sentential	The numbers, 1, 2, 3, 4, 5 may be said in order
6	Preoperational	The objects in row 5 may be counted. The last count called 5, five, cinco, etc. * * * * * ■ ■ ■ ■ ■ ○ ○ ○ ○ ○ ■ / ○ ▣ □
7	Primary	There are behaviors that act on such classes that we call simple arithmetic operations $1 + 3 = 4$ $5 + 15 = 20$ $5(4) = 20$ $5(3) = 15$ $5(1) = 5$
8	Concrete	There are behaviors that order the simple arithmetic behaviors when multiplying a sum by a number. Such distributive behaviors require the simple arithmetic behavior as a prerequisite, not just a precursor $5(1 + 3) = 5(1) + 5(3) = 5 + 15 = 20$
9	Abstract	All the forms of five in the five rows in the example are equivalent in value, x = 5. Forming class based on abstract feature
10	Formal	The general left hand distributive relation is $x \times (y + z) = (x \times y) + (x \times z)$
11	Systematic	The right hand distribution law is not true for numbers but is true for proportions in logic. $x + (y \times z) = (x + y) \times (x + z)$ $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$ $x \,\&\, (y \text{ or } z) = (x \,\&\, y) \text{ or } (x \,\&\, z)$
12	Metasystematic	The system of propositional logic and elementary set theory are isomorphic $x \,\&\, (y \text{ or } z) = (x \,\&\, y) \text{ or } (x \,\&\, z)$ Logic $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$ Sets T(False) φ Empty set T(True) Ω Universal set