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Hierarchical Complexity of Tasks Shows the Existence of Developmental Stages

Michael Lamport Commons

Harvard Medical School

Edward James Trudeau and Sharon Anne Stein

Harvard University

Francis Asbury Richards

Cornell University

and

Sharon R. Krause

Harvard Divinity School

The major purpose of this paper is to introduce the notion of the order of hierarchical complexity of tasks. Order of hierarchical complexity is a way of conceptualizing information in terms of the power required to complete a task or solve a problem. It is orthogonal to the notion of information coded as bits in traditional information theory. Because every task (whether experimental or everyday) that individuals engage in has an order of hierarchical complexity associated with it, this notion of hierarchical complexity has broad implications both within developmental psychology and beyond it in such fields as information science. Within developmental psychology, traditional stage theory has been criticized for not showing that stages exist as anything more than *ad hoc* descriptions of sequential changes in human behavior

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Address reprint requests to Michael L. Commons, Department of Psychiatry, Harvard Medical School, Massachusetts Mental Health Center, 74 Fenwood Road, Boston, MA 02115-6196.

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(Kohlberg & Armon, 1984; Gibbs, 1977, 1979; Broughton, 1984). To address this issue, Commons and Richards (1984a,b) argued that a successful developmental theory should address two conceptually different issues: (1) the hierarchical complexity of the task to be solved and (2) the psychology, sociology, and anthropology of such task performance and how that performance develops. The notion of the hierarchical complexity of tasks, introduced here, formalizes the key notions implicit in most stage theories, presenting them as axioms and theorems. The hierarchical complexity of tasks has itself been grounded in mathematical models (Coombs, Dawes, & Tversky, 1970) and information science (Lindsay & Norman, 1977). The resultant definition of stage is that it is the highest order of hierarchical complexity on which there is successful task performance. In addition to providing an analytic solution to the issue of what are developmental stages, the theory of hierarchical complexity presented here allows for the possibility within science of scaling the complexity in a form more akin to intelligence.

Key Words: Existence; development; stages; hierarchical; complexity; task-analysis; analytic; sequence; cognitive; Piaget.

The major purpose of this paper is to introduce the notion of the order of hierarchical complexity of tasks. Hierarchical complexity describes a form of information that is orthogonal to the traditional information theory form, in which information is coded as bits that increase quantitatively with more information. Our proposal provides an analytic measure of the power required to complete a task or solve a problem.

This conceptualization establishes the notion of hierarchical complexity as a new branch of information science with broad implications both within and beyond the confines of developmental psychology. Because hierarchical complexity is such an ever-present dimension of tasks, taking it into account will make certain behavioral science issues more coherent and our analysis of them more powerful and effective. This is because every task has an order of complexity associated with it. This means that within behavioral science every experimental task, every clinical test, developmental task, survey item, or statement by a person can be characterized in terms of its hierarchical complexity. Other tasks and activities can be similarly classified; for example, jobs and activities, political systems, or economic systems. Measures that ignore the hierarchical complexity of tasks collapse the performances obtained in ways that obscure the factor(s) that are actually causing the variability in behavior. For example, one speculation is that as individuals in given societies get more educated, class status is due less to education per se and more to parental status, income, and occupation. This might be because the hierarchical complexity of the tasks a person solves determines income now more than education. Few can meet the highest demands or can solve the most hierarchically complex tasks; quite a few can meet only the minimal demands. Outside the behavioral sciences, the notion of hierarchical complexity may turn out to be useful. For example, it may provide a way to organize the functions of large numbers of processes within the biological sciences, evolutionary data, or to measure the power of computers, robots, or programs within the computational sciences.

We first present the historical background of stage within developmental psychology. We will discuss why hierarchical complexity is useful within developmental psychology. The first problem that it addresses is the existence of developmental stages. The traditional notion of developmental stages has been hopelessly mired in empirical problems that have led to its abandonment by many as a coherent measure (see discussions by Brainerd, 1978, and Broughton, 1984). Here, we replace the old, empirically based, idea of developing mental structures with an analytic notion—that of the hierarchical complexity of tasks. We demonstrate further that the identical measures of hierarchical complexity underlie development in any domain.

The difficulty of empirically demonstrating the existence of stage has a long history in developmental research. Traditional stage theory has been criticized for not showing that stages exist as more than *ad hoc* descriptions of sequential changes in human behavior (Kohlberg & Armon, 1984; Gibbs, 1977, 1979; Broughton, 1984). Fischer, Hand, and Russell (1984), along with Case (1985), have demonstrated the problems of confounding developmental sequence of behavior with traditional notions of stage in the quest for empirical evidence. Sequential acquisition of behavior can clearly be demonstrated empirically. A precise and defensible notion of stage has proven more elusive. An empirical test of some notion of stage, as Campbell and Richie (1983) have made clear, would involve a demonstration that qualitative differences exist between one stage and the next one. We refer to such qualitative differences as discreteness or gappiness. Such flawed demonstrations have almost always been empirical. Our notion of ‘‘stage’’ introduced here is based on hierarchical complexity of tasks and then on performance on those tasks. Our notion does not require in any way abrupt emergence of the new performance and abrupt displacement or disappearance of the old performance. Maybe intervention studies could be used to show this discreteness, but even these kinds of studies cannot prove the existence of some form of empirically defined stage (Commons & Calneck, 1984). The problem is like the one of ‘‘all or none learning.’’ If one makes measurements continuously, then one may find continuity in acquisition—or one may find periods of no change and apparent instantaneous change. Whether the change appears discontinuous depends upon how often one makes the measurement. This position is consistent with many of the newer transition models in which change may take place without the behavioral markers changing. Molenaar and van der Maas (1994), and van der Maas (1992) and their collaborators use mathematical models in an attempt to show how changes in an underlying continuum could produce abrupt transitions. If one measures less often, one sees jumps in performance or gaps between subject performance measured at one time. If one measures more often one may see what appears more like continuous acquisition. The current model deals with this problem by uncoupling the measured or empirical performance from the two components upon which the performance is based: the hierarchical complexity of the task on the one hand and additional factors that influence performance on the other.

Despite the difficulties associated with empirical accounts of a notion of stage development, a post-Piagetian, purely analytic, basis for a notion of stage has not yet been set forth. The aim of this paper is to do exactly that—to demonstrate the existence of stage *a priori*. Because of its analytic nature, the new notion of stage jettisons many of the assumptions of traditional stage models. The General Model of Hierarchical Complexity (GMHC) introduced here axiomatically defines what we suggest are the minimum necessary requirements for the existence of developmental stages (for an earlier version, see Commons & Richards, 1984a,b). The term ''axiom'' as used here refers to foundational statements that are acceptable without proof and cannot fruitfully be defined further. To show that there are groups of tasks that satisfy the axioms of the General Model of Hierarchical Complexity will demonstrate the existence of stages and stage sequences analytically rather than empirically.

THE GENERAL MODEL OF HIERARCHICAL COMPLEXITY

At this point, we will briefly present the main elements of the General Model of Hierarchical Complexity. Note that the notion of hierarchical complexity of tasks is grounded in mathematical models (Coombs, Dawes, & Tversky, 1970) and information science (Lindsay & Norman, 1977). By distinguishing the notion of subject performance from that of task demand (Desberg & Taylor, 1986; Taylor, 1903/1911) our model grounds the definition of developmental stage in the hierarchical complexity of tasks (see Frege (1960, 1964) for application to mathematics).

Commons and Richards (1984a,b) emphasized that developmental theory should address two conceptually different issues: (1) the hierarchical complexity of the task to be solved and (2) the psychology, sociology, and anthropology of how such task performance develops. The General Model of Hierarchical Complexity uses the hierarchical complexity of tasks as the basis for the definition of stage. An action is at a given *stage* when it successfully completes a task of a given hierarchical order of complexity. Roughly, *hierarchical complexity* refers to the number of nonrepeating recursions that the coordinating actions must perform on a set of primary elements. Actions at a *higher order of hierarchical complexity:* (a) are *defined* in terms of the actions at the *next lower* order of hierarchical complexity; (b) *organize* and *transform* the lower order actions; (c) produce organizations of lower order actions that are new and *not arbitrary* and cannot be accomplished by those lower order actions alone.

In addition, the model describes discrete orders of hierarchy of task complexity, sets forth a set of axioms that have to be satisfied in order to define a stage sequence, and describes the necessary analytical properties of hierarchical orders. It does not posit detailed empirical forms of stages or the empirical processes that cause stage change. Nor does the existence of stages or of stage sequence shown through task analysis depend on any particular psychological assumptions about stage or stage change (e.g., Case, 1985; Fischer, 1980; Rosales & Baer, 1997). With tasks as the fundamental elements to be measured, task analysis and the sequential ordering of tasks are alone sufficient to form a ‘‘stage’’ sequence. From the order of hierarchical complexity of tasks, then, it is possible to derive an analytic measure of stage.

This paper further describes the axiomatic criteria for a theory to be accepted as a stage theory. The fact and sufficiency of the current proposed analytic model can increase accuracy and consistency in measuring stage of development and will establish greater coherence in the field of developmental theory, as well as in other areas of psychological study.

DEVELOPMENT OF STAGE THEORY

In order to establish what the minimum criteria are for a stage theory, we will next briefly review earlier theories. Throughout most of this discussion we will rely on the terms stage and stage sequence as those have been most prevalent in the literature. Predating the line of research that has led to modern stage theories of human development, Frege (1879/1967) expounded a stage sequence for a part of mathematics (see Table 1 for a more complete version). Some of the earlier and most influential psychological theories of human development (Binet, 1905/1916) used maturation over time as the measure of behavioral change. While these theories focused on the duration of time of developmental periods, some early theorists (Baldwin, 1906) tried to categorize these periods into sequences or levels. Piaget (1937/1954) was one of the first theorists to focus on patterns of behavioral change. Since his formalized stages of development in children were published, stage theory has been a dominant force within developmental theory. It is worth noting that Piaget’s aim in introducing stages of performances on tasks was always a taxonomic one. Stages were a means of classifying instances of thinking shown while working on some tasks. In some of his writings in the 1920s, Piaget would even classify different protocols from *the same child* (at the same time of testing) at different stages. The taxonomic theme predates Piaget’s interest in developmental psychology, as presented carefully and convincingly by Chapman (1988). The notion of hard stages, while in part abstracted from Piaget, is directly articulated by Kohlberg, one of the first to bring Piaget’s work to the attention of a large number of psychologists and educators.

In the General Model of Hierarchical Complexity, the classic Kohlberg

TABLE 1
Sequence of Orders of Complexity for Distributivity in Arithmetic

Order of Hierarchical Complexity	Name	Example
0	Calculatory	Simple machine arithmetic on 0's and 1's
1	Sensory and motor	Seeing circles, squares etc. or touching them.
2	Circular sensory–motor	Reaching and grasping a circle or square. * * * * * ○ ○ ○ ○ ○ ■ ■ ■ ■ ■ ■ / ○ ▣ □
3	Sensory–motor	A class of filled in squares may be made
4	Nominal	That class may be named ''Squares''
5	Sentential	The numbers 1, 2, 3, 4, 5 may be said in order
6	Preoperational	The objects in row 5 may be counted. The last count called 5, five, cinco etc.
7	Primary	There are behaviors that act on such classes that we call simple arithmetic operations $1 + 3 = 4$ $5 + 15 = 20$ $5(4) = 20$ $5(3) = 15$ $5(1) = 5$
8	Concrete	There are behaviors that order the simple arithmetic behaviors when multiplying a sum by a number. Such distributive behaviors *require* the simple arithmetic behavior as a prerequisite, not just a precursor $5(1 + 3) = 5(1) + 5(3) = 5 + 15 = 20$
9	Abstract	All the forms of five in the five rows in the example are equivalent in value, $x = 5$. Forming class based on abstract feature
10	Formal	The general left hand distributive relation is $x * (y + z) = (x * y) + (x * z)$
11	Systematic	The right hand distribution law is not true for numbers but is true for proportions and sets. $x + (y * z) = (x * y) + (x * z)$ $x \cup (y \cap z) = (x \cap y) \cup (x \cap z)$
12	Metasystematic	The system of propositional logic and elementary set theory are isomorphic x & (y or z) = (x & y) or (x & z) Logic $\Leftrightarrow x \cap (y \cup z) = (x \cap y) \cup (\mathrm{x} \cap \mathrm{z})$ Sets T(False) $\leftrightarrow \phi$ Empty set T(True) $\leftrightarrow \Omega$ Universal set

and Armon (1984) interpretation of ‘‘Piagetian stage conditions’’ are met and surpassed:

Condition 1. ‘‘Qualitative differences in stages’’ is shown in a General Model of Hierarchical Complexity theorem—Qualitative differences are the opposite of the Archimedean principle that between any two adjacent ordinal numbers one can find another number.

Condition 2. ‘‘Invariant sequence’’ is shown by a theorem that follows from the definitions.

Condition 3. ‘‘Structure-of-the-whole’’ is an axiom on tasks and is true everywhere in every domain and content.

Condition 4. ‘‘Stages are hierarchical integrations’’ is fulfilled by definition.

We add:

Condition 5. ‘‘Logic of each stage is explicit’’ so that the sequence of stages can be tested analytically and new tasks can be classified systematically. This condition is also met by the General Model of Hierarchical Complexity.

We interpret these conditions as follows. In Condition 1, stages imply a qualitative distinction in mathematical–logical properties of the tasks that actions address. Those actions must still serve the same function at various points in development. In Condition 2, stages form an invariant sequence underlying the course of individual development. In Condition 3, these stages exhibit the structure-of-the-whole property. The same stage of performance is demanded across all domains. The phenomenon of unequal development of performance in different domains is termed *décalage*—unequal performance usually being the rule. People’s highest stage response characterizes the most hierarchically complex task they solve. Their action corresponds to the *stage structure*, which is the underlying organization of thought and actions demanded by a task. Our model rejects *décalage* of ‘‘stage structures,’’ but not that of resulting performance. In Condition 4, the stage sequence is hierarchical, with the higher stages integrating and transforming the lower.

Although of descriptive value, the Kohlberg and Armon (1984) ‘‘Piagetian’’ requirements conflate analytic and empirical criteria and thus have not engendered a universally accepted formal analysis of stage. The General Model of Hierarchical Complexity separates the ‘‘stage’’ properties for performance from the tasks they address. The model also adds a fifth condition, which is that the logic of each stage’s tasks must be explicit so that the sequence of stages can be tested analytically and new tasks can be classified systematically. This condition makes for an even ‘‘harder’’ stage model than that required by the original four hard-stage conditions. Kohlberg and Armon (1984) have noted the current disarray in the field and the resulting theories of ‘‘soft’’ stages that fit some but not all of Piaget’s criteria.

Since Piaget’s death, few absolute and systematic accounts of stage have been devised, and the notion of stage itself has increasingly come under attack (Brainerd, 1978; Broughton, 1984; Flavell, 1963). The diversity of opinion as to the existence and measure of stage may be grouped into eight broad categories, four of which reject and four of which accept some notion of stage (for detailed reviews of some present stage theories, see for example Alexander & Langer, 1990; Campbell & Bickhard, 1986).

First, the concept of stage has been rejected categorically by some behavioral psychologists, most notably Skinner (Skinner & Vaughan, 1983). Behaviorists acknowledge that certain behaviors require the acquisition of others in a chronological sense, but do not characterize these sequences in any definitive way. The emergent property of stage (see Table 4) that Piaget posits is particularly objectionable (Gewirtz, 1991). It assumes automatic generativity rather than the contextualist notion that development is produced by an interaction between actions and the surrounding context.

Second, there are those theorists focusing on maturation and IQ studies (Binet & Simon, 1905/1916; Gesell & Amatruda, 1964; Terman & Merrill, 1937; Wexler, 1982) who work with the notion of sequence rather than stage. To the behaviorists' notion of chronological acquisition of behaviors, these theorists add a requirement of certain ages for certain acquisitions, thereby charting human development in a normative way that allows for averages and deviancies.

Third, the work of those who characterize development in terms of periods, or ''seasons'' in human life, rejects the concept of stage. Among some theorists, these periods are quite specialized (Erikson, 1959, 1978, 1982; Levinson, 1986), although among others they form only three or four broad superperiods (Alexander, Druker & Langer, 1990; Flavell, 1963). These periods are seen as sequenced but not hierarchical, and they are not organized in any strictly logical way. Development in terms of periods may be characterized more as socialization, whereas stage development is understood as transformation.

Fourth, there are some who characterize development in terms of levels, reserving the notion of stage to refer to coherence across domains within a chronological period (Case, 1974, 1978, 1982; Fischer, 1980; Pascual-Leone, 1970, 1984). These theorists accept *décalage* just as Piaget (1972) did and maintain that functioning at different levels in different domains is normal. These levels are hierarchical, but they may lack a single reflective, generative mechanism, which characterizes a stage. Depending on the level, different kinds of coordinations may take place. These differences are often organized into tiers containing four levels. The first level of one tier parallels the first levels of other tiers. These levels fail to integrate and order lower stage actions in the same way irrespective of level. Additionally, Campbell and Bickhard (1986) claim that level theories (e.g., Case, 1978, 1982; Fischer, 1980) do not build upon a reflective abstraction process of stage change that underlies the whole process of development. Thus, the levels tend to have the qualities of subroutines, so that one cannot be sure that these levels could not be grouped into larger routines or subdivided even further (see Campbell & Bickhard, 1992) because of the lack of inherent hierarchy.

Among those who do accept the notion of stage, a great deal of controversy exists as to the nature and measurement of these stages. First among these is the view, held by many stage theorists, that human beings move through developmental stages, but that this development stops at the stage of formal operations or shortly thereafter (Baltes, 1987). There is also diversity among those who accept the notion of adult stages (for a discussion of these adult stage theories, see Alexander & Langer, 1990; Kohlberg & Armon, 1984; and Richards & Commons, 1990a,b). Piaget made a number of clear statements about postformal operations. Campbell (personal communication) reports Piaget's statement that constructing axiomatic systems in geometry requires a level of thinking that is a stage beyond formal operations: ‘‘one could say that axiomatic schemas are to formal schemes what the latter are to concrete operations'' (Piaget, 1950).

A second perspective on stage theory can be identified with the work of Loevinger, Kitchener, and King or others who work with stage in a statistical sense and use psychometric methods to support their theories. Among these theorists (Kitchener & King, 1990; Loevinger & Blasi, 1976; Rest, Turiel, & Kohlberg, 1969) one does not find a clearly delineated *a priori* logic of stages (Kohlberg, 1984; Kohlberg & Armon, 1984).

A third alternative is found in the work of Armon (1984), Dasen (1977), Kegan (1982), Kohlberg (1984), and Selman (1980), who define clear stages, but whose theories of stage self-admittedly lack a solid foundation in logic.

A fourth alternative has been to extend accounts with organized schemata underlying each stage (Demetriou, 1990; Demetriou & Efklides, 1985; Pascual-Leone, 1984). This fourth alternative has generated a structure and sequence of formal and postformal thought. Kohlberg (1990), Demetriou (1990; Demetriou & Efklides, 1985), and Pascual-Leone (1984) all conceived of two to three postformal stages and saw the necessity of logical analysis to demonstrate and clarify these stages (Commons & Grotzer, 1990). That is, Kohlberg, for example, said that stage is a logical truth and that postformal stages existed (Kohlberg, 1990). This paper continues that one aspect of Kohlberg's work by formalizing that truth in an analytic model.

Finally, a few stage theorists have provided an analysis of the constraints on stage, such as Campbell and Bickhard (1986, 1992). These theorists clearly accept the notion of adult stages and provide a logical psychological analysis of these stages. The difficulty with their model is that it neglects the task analysis that would support their psychological claims. Campbell and Bickhard are correct in rejecting task analysis as a sufficient solution to the riddle of stages, but their logical analysis cannot tell us what representations are actually functioning in a given task. A *levels* account such as Campbell and Bickhard's does not equal a detailed account of representations.

More empirical work on this model would shed light on some of the difficulties of Campbell and Bickhard's stages of reflective abstraction. For us, their model seems to require too advanced a level of consciousness to account sufficiently for the lower portions of the sequence. They argue (personal communication) that, from the standpoint of the GMHC, Campbell and Bickhard's Knowing Level 1 certainly appears to span too many stages. Nonetheless, they really did mean it when they said Knowing Level 1 starts at birth. They ask, ''What could precede interactive knowledge that is based on direct interaction with the environment?'' Table 2 shows some of the more common stage systems and their rough equivalencies.

ANTECEDENTS TO HIERARCHICAL TASK COMPLEXITY AS THE BASIS FOR ANALYTIC STAGE MEASURES OF BEHAVIOR

The purpose of this next section is to discuss some forerunners to the present GMHC. Newell, Shaw, and Simon (1964) developed a general information-processing theory. Using information-processing theory, Case (1985), Pascual-Leone (1970, 1984), Scandura (1977), and Siegler (Siegler, Liebert, & Liebert, 1973; Siegler, 1981) applied the role of task and task analysis giving tasks a central role in the conception of stage. However, task analysis (and information-processing models of stage) have not stated the *a priori* analytic requirements of a hierarchical model of tasks. Historically, two methods for determining the stage of subject performance (beginning at the preschool level) on tasks in a particular domain have predominated (Feldman, 1980). The first method is the observation/interview method that uses a single task to test for multiple stages; the second method is the observation/interview on a series of tasks, or task sequence method, where each task is created to test just one stage. Both methods inductively infer stage of performance on a single task in a domain.

With the observation/interview method (Inhelder & Piaget, 1958; Kegan, 1982; Kohlberg, 1981; Selman, 1980; Piaget, 1927/1930) the researcher presents the subject with a task and then interviews the subject to verify the successful completion of the task. The observation/interview method was at the heart of Piaget's enterprise. From childrens' performances, Piaget inferred the stage at which they ''constructed'' the task. The subjects' constructions of the task are the actions, both internal and external, that they perform with respect to the task. To assign a stage to a subject's performance from an examination of the subject's actions, one must assign a stage to the task that one infers the subject has solved. When subjects fail to solve the task at hand, the scoring scheme, in order to specify which lower stage the subject's solution represents, must indicate which, if any, lower stage task that solution would successfully address. In the case of the observation/interview method, the task that is being adequately solved is often inferred by researchers.

TABLE 2
Relationships among Stage Models

	GMHC	Fischer Tier	Fischer Level	Kohlberg, Armon, Selman	Case	Campbell, Bickhard	Piaget	Approx. age
Computory	0							
Sensory and motor	1		0	−1	2	1		0–0.5
Circular sensory-motor	2	I	1	−1/0	3		a	0.5–1
Sensory-motor	3		2	0	4		b	1–2
Nominal	4		3	0/1	5		Ia	2–3
Sentential	5		3–4	1	5–6			
Preoperational	6	II	4	1/2	6		Ib	4–6
Primary	7		5	2	7	2	IIa	6–8
Concrete	8		6	2/3	8		IIb	8–10
Abstract	9	III	7	3	9	3	IIIa	10
Formal	10		8	3/4	10		IIIb	12
Systematic	11		9	4	11	4	IIIc	18+
Metasystematic	12	IV	10	5–6	12		Postformal	20+
Paradigmatic	13		11	6	?	5	Postformal	?
Cross-paradigmatic	14		12	7	?		Postformal	?
	15		non-existent	8			Not observed	?

With the task sequence method, on the other hand, the researcher deductively determines the stage of the task by analyzing a series of tasks in a domain that the subject has successfully completed (Case, 1985; Commons & Richards, 1984b; Fischer, 1980; and Siegler, 1978, 1981). In the task sequence method, the task demands are deductively determined by the researcher or theorist beforehand. If the subject performs the task successfully, one knows directly that the subject's performance meets the requirements of that task. When one knows the task demands and the stage that they require, it is not necessary to infer the stage of the subject's construction of a successfully completed task. Additionally, there is the requirement that the subject be presented with an explicit sequence of tasks covering each of the orders in the given domain (Fischer, 1980; Commons & Richards, 1984b). Otherwise one would only know whether the performance was at least at the stage required by the task or below that stage without analysis of the verbal responses.

In both cases, however, the subject response is quite determined by the hierarchical complexity of tasks adequately solved. In the observation/interview method, the subject's solution to the task meets a set of task demands characteristic of that stage. In the task sequence method, the stage of performance required by the given order of task complexity has been determined before the subject addresses the task. One problem with this method is that it does not usually permit subjects to construct tasks on their own. The construction problem increases the difficulty of a task. For example, Kuhn, Amsel, O'Loughlin et al. (1988) found that fewer college students were scored as formal operational on self-constructed tasks than when college students are administered formal operational tasks constructed by the researchers (e.g., Commons, Miller, & Kuhn, 1982). In any case, an analysis of task complexity is necessary for both methods in order to determine the stage of any given solution to the task. Because performance always involves a particular task, descriptions of subject performance that lack task analysis are bound to be incomplete at best and at worst erroneous.

An indication of where ''stages'' exist is essential to the construction of an analytic measure of order of hierarchical complexity. It is not immediately obvious, however, whether stages are supposed to be properties of people, performances, tasks, interactions, or even the world at large. Many researchers consider stage to be an epistemological competence internal to a subject that is separable from performance. Commons and Richards (1984b), however, claim that whether or not there is an internal epistemological competence, one can only assess the stage of response required by a given task. Commons and Richards contend that what is measured is performance on a task of analytically determined hierarchical complexity. The task may be created at a given order of hierarchical complexity or inferred as the task the participant addresses. There is no such thing as competence.

Observable interaction between the researcher and the participant is always grounded in an ideal, the ideal here being hierarchical complexity of task. To paraphrase Kant (1781/1965) from the *Critique of Pure Reason*, observation without concepts is blind, and concepts without observation are empty. According to the General Model of Hierarchical Complexity, therefore, stage is a property of subject behavior, or response. All behavior of a participant has a stage with respect to the task. Specifically, stage characterizes a subject response to the effective stimuli of given hierarchically ordered task demands. The General Model of Hierarchical Complexity is constructed entirely from the observer's perspective. It consistently avoids referring to the organism's knowledge or perspective in its characterization of order of complexity.

Hierarchical complexity is a mathematical concept. Most modern theories are written in the language of mathematics to improve the definiteness of statements and reduce excess meaning. Just as the truth value of mathematical equations is independent of any individual's ability to perform them, so the analytic notion of stage does not refer to actual participants' performance. The stage of response required by a task exists irrespective of how real people perform on the task. Consider the expression, $1 + 1 = 2$, with the numeric symbols standing for quantities without reference to what these quantities are of, or to people doing the task, or the objects added. The response required to solve this addition problem is primary stage, regardless of whether actual subjects meet the requirement (and solve the problem). Piaget referred to this stage as early concrete operations. Here the stage refers only to a point along a sequence, not some shared structure-of-the-whole.

TASK ANALYSIS

It is a fundamental assumption that tasks can be analyzed in terms of the actions that they require for successful completion. More than one set of actions may complete a task. A proof of the analysis is to carry out the actions and see if the task is completed. These actions are called *task-required actions* or *task demands,* or just demands. For example, a computer might add the number 1 to the number 10, producing the number 11. *Task*, by turn, is defined as some ''ideal'' set of actions that are performed on some ideal set of objects. The *task demands* are the contingencies between ideal behavior and stimuli in the situation. We define *contingency* very broadly as a relationship between events (i.e., behaviors or responses) and outcomes. The development of the fields of artificial intelligence and information processing have made it possible to list the series of actions involved in a given task. This type of protocol analysis illustrates that the logical assembly of a given set of actions will result in the desired outcome on a particular task. There is no task having an infinite number of actions that a human being can perform in a finite period of time. Conversely, there is a limit to how short a measurable action can be. This probably makes it possible to analytically decompose most tasks that exist but maybe not all. At the heart of the model is the notion of hierarchical (vertical) complexity—as opposed to nonhierarchical (horizontal) complexity. First we review nonhierarchical complexity to clarify the difference between the two forms of information processing.

HIERARCHICAL VERSUS NONHIERARCHICAL COMPLEXITY OF TASKS

Information in tasks and their subtasks can be characterized by both hierarchical (vertical) and nonhierarchical (horizontal) complexity. *Hierarchical complexity* is the order of hierarchical complexity of the task (Commons & Richards, 1984a; Commons & Rodriguez, 1990, 1993). *Nonhierarchical complexity* refers to the classical notion of complexity found in information theory (Shannon & Weaver, 1949).

NONHIERARCHICAL COMPLEXITY

Here, for every yes–no question, there is by definition an answer containing one bit of information. Each additional yes–no question adds another bit of information. The amount of such information required to solve a task determines its nonhierarchical complexity. For example, in determining the amount of time a computer program requires to execute a program, the number of bits yields an estimate. The measure of how many simple nonrecodable items an organism can work with at a given moment is given in terms of the number of bits. Difficulty is often a property of nonhierarchical complexity. Doing 20 addition problems is of the same order of complexity as doing 1 addition problem, for example. The number of concatenation operations in 20 addition problems is the same as the number of concatenations in a single addition problem. However, most subjects would agree that a task that includes 20 problems is more difficult than one which includes only 2, because it requires more work. Such nonhierarchical demands may include high memory requirements, iterative problem solving involving time-consuming rote work, lack of familiarity with the elements of a problem, lack of available representations and codes for those representations, and so on. It is important to note the difference between hierarchical and nonhierarchical complexity, because a subject could fail the most difficult order-three task, but complete the simplest order-four task. This may lead to uncertainty or ambiguity in scoring the performance: Can the subject really perform a problem with order four of hierarchical complexity in the domain in question? In such a case, researchers should consider the nonhierarchical complexity of the task and whether nonhierarchical task demands may have made the order-three task especially difficult for this subject.

The number of bits of information needed to complete a task is described in information theory. By definition, for every yes–no question embedded in a task, there is an answer containing one bit of information. Take the simplest situation, in which a yes/no question is transmitted. If one has one yes–no question, the answer (by definition) contains 1 bit of information ($n = 1$). To see this, consider that for two yes–no questions, there are 2 bits of information, and there are $4 = 2^2$ alternative answer patterns (yes–yes, yes–no, no–yes, no–no). For three questions, there are 3 bits of information and $8 = 2^3$ alternatives. In any situation made up of a number of such yes–no questions, the number of alternative answers can be generated by calculating 2^n, where n is the number of bits. If there are n questions, each one containing 1 bit of information, there are 2^n alternative answers. The amount of this type of information required by a problem is called the *horizontal* complexity. *Horizontal* complexity may account for differences in difficulty of tasks but not for differences in the order of hierarchical complexity.

HIERARCHICAL COMPLEXITY

In the model presented here we are concerned with hierarchical complexity only, not horizontal complexity. For example, multiplying $3 \times (9 + 2)$ requires a distributive action at the concrete stage of hierarchical complexity. The distributive action is as follows: $3 \times (9 + 2) = (3 \times 9) + (3 \times 2) = 27 + 6 = 33$. That action coordinates (organizes) adding and multiplying by uniquely organizing the order of those actions. The distributive action is therefore one order more complex than the acts of adding and multiplying alone. Although one could arrive at the same answer through simple addition, performing both addition and multiplication in a coordinated manner facilitates greater effectiveness in problem-solving. The distributive action forms a pattern out of the additive and multiplicative actions. Traditional information processes in computers do not do pattern recognition and detection well in general but simple distributivity might be recognized.

The Order of Hierarchical Complexity or just order of tasks is determined by the number of nonrepeating recursions which constitute it. *Recursion* refers to the process by which the output of the lower order actions forms the input of the higher order actions. The order of hierarchical complexity of task T is denoted $C(T)$ and defined as follows:

(1) For a simple task t_i, $C(t_i)$ is 1.
(2) Otherwise, $C(T) = C(T') + 1$,
where $C(T') = \max(C(T_1, C(T_2, \ldots C(T_n))$ for all T_i in T.

In parallel with the discussion about horizontal information theory, for every additional coordination (organization of actions) there is one more order (which is parallel to the notion of bits in horizontal information processing theory). There are at least two actions from the next lowest stage coordinated, so order number is n and the number of actions coordinated is 2^n.

A *simple task* has complexity one, and all other tasks have complexity one greater than the complexity of their highest task demand. Because grabbing an object requires glancing and touching, both simple tasks of complexity one, the complexity of grabbing is the max (complexity of glancing, complexity of touching) plus one, which is two. The hierarchical complexity of glancing is one, the hierarchical complexity of grabbing is two. The theory of task analysis prohibits a task of order *n* from being a prerequisite to a task of order less than *n*. Glancing can be an element of grabbing, but grabbing cannot be an element of glancing, although some tasks may overlap in their demands, such as grabbing an object and hitting an object, which both require glancing at an object. Every task (unless it is a simple task) therefore contains a multitude of subtasks. This ‘‘nesting’’ of lower order tasks within higher order tasks is called *concatenation.* Each new task-required action in the hierarchy is one order more complex than the task-required actions upon which it is built. Tasks are always more hierarchically complex than their subtasks. In this task-complexity theory, for a task to be more hierarchically complex than another, the new task must meet three requirements. Again, the new task-required action must

(1) be *defined* in terms of the lower stage actions and
(2) *coordinate* the lower stage actions in a
(3) *nonarbitrary* way.

First, the very definition of a task-required behavior with a higher complexity must depend on previously defined, task-required behavior of lower complexity. Second, the higher complexity task-required actions must coordinate the less complex actions. To coordinate actions is to specify the way a set of actions fit together and interrelate. The coordination specifies the order of the less complex actions. Third, the coordination must not be arbitrary. Otherwise the coordination would be merely a chain of behaviors. The meaning of the more complex task must be severely altered by any nonspecified alteration in the coordination.

Through such task analysis, the hierarchical complexity of a task may be determined. The hierarchical complexity of a task therefore refers to the number of concatenation operations it contains. An order-three task has three concatenation operations. A task of order three operates on a task of order two and a task of order two operates on a task of order one (a simple task). Figure 1 shows a sample task tree.

The present model allows comparisons of sequences of tasks across domains, without invoking horizontal structure or making developmental synchrony claims. There is no good taxonomy of collections of tasks. The Piagetians use the word domain, such as in the moral, or physical domain. Campbell (personal communication) suggests the word field would be more appropriate because it refers to the task side rather than performance side. Although we use the term *domain,* which reflects the organism's perspective, because it is common, we use it to mean the external-perspective term *field.* In ordering unrelated tasks, one starts with a set of related tasks. The related task sequence should be large enough to contain examples of the entire sequence of orders of complexity. Finding tasks that span the whole sequence

Addition

1\. $e_{+}(a_i) \longrightarrow e_{1+}(1,2,3) \longrightarrow 1+2=3$

$e_{2+}(1,3,4) \longrightarrow 1+3=4$

$e_{3+}(2,3,5) \longrightarrow 2+3=5$

$e_{4+}(3,6,9) \longrightarrow 3+6=9$

Multiplication

2\. $e_{x}(a_i) \longrightarrow e_{1x}(1,2,2) \longrightarrow 1x2=2$

$e_{2x}(1,3,3) \longrightarrow 1x3=3$

$e_{3x}(2,3,6) \longrightarrow 2x3=6$

$e_{4x}(3,3,9) \longrightarrow 3x3=9$

Distribution

3\. $e_{d}(a_i\, a_j) \longrightarrow e_{1d}(1,2,3) \longrightarrow 1x\ (2+3) \longrightarrow (1x2) + (1x3) = 2 + 3 = 5$

$e_{2d}(1,3,3) \longrightarrow 1x\ (3+3) = (1x3) + (1x3) = 3 + 3 = 6$

$e_{3d}(2,3,6) \longrightarrow 2x\ (3+6) = (2x3) + (2x6) = 6 + 12 = 18$

$e_{4d}(3,3,9) \longrightarrow 3x\ (3+9) = (3x3) = (3x9) = 9 + 27 = 36$

FIG. 1. Sample task tree.

is not difficult because a very large number of sequences are known. Also, descriptions exist for each order of hierarchical complexity. To determine the hierarchical complexity of a task from a new sequence, one follows the same procedure as used to analyze the early task sequences. One finds some complex versions of the task. One breaks down the task step by step. One answers the question at each order of hierarchical complexity, what less complex actions is the required task action ordering? One then tries to see if there are intermediate actions.

TASK CHAINS

Behavior can be sequenced so that, to meet the requirements of the world, some behaviors have to be executed before others. For example, in the present, one has to press the elevator button before the elevator door opens. One has to wait until the door is open before walking onto the elevator. This task chain is ordered by steps that one can imagine could be eliminated. Someone else could have pushed the button or a sensor could determine that you have come up to the elevator. These actions are just *predecessors* of actions. They go before, without being necessary to the following action. They are not prerequisites in a hierarchical sequence. They come first but the order of execution or acquisition could be altered.

In contrast to the above example, hierarchical task-chain sequences are organized and this organization is not arbitrary. There are powerful implications for the nature of the ordering that hold for stacked neural nets (the output of one group of neural nets feeding into another net—with feedback) and animals. Usually lower order actions must be executed before it is possible to execute a higher order action. Again, a *chain* is a sequence of tasks such that a task in the chain goes either before or after another task. In a hierarchical task chain, each action includes the task below or before it. A task is higher on the chain if it requires another task in the chain to complete it. The lowest task on the chain has no prerequisite in the chain, and the highest task on the chain is not a prerequisite for any task on the chain. For example, in order to count one must say numbers in sequence; in order to say numbers in sequence one must be able to say numbers. Here, saying numbers is the lowest task on the chain, and counting is the highest task on the chain. Two tasks are called *distinct* just in case one is not the inverse or identity of the other. Saying ''one'' and saying ''one'' a second time are two instances of the same task. Not every action has an inverse, although some actions have multiple inverses. For example, some problems in mathematics have inverses that are comparable to an additive inverse, while others have multiplicative inverses.

Two or more distinct actions will form a nonarbitrary organized chain if the outcome of one depends upon the outcome of the other. This means that the outcome of the first serves as the input for the second. The preceding action is called a *prerequisite action.* Grabbing a pen depends on successfully glancing at a pen and successfully touching a pen, so glancing at a pen and touching a pen are prerequisite actions for grabbing a pen. A higher-order, organizing action is necessary. An *organizing* action relates two or more lower order tasks in a chain. Grabbing organizes glancing and touching. Tasks whose chains intersect on some significant number (as defined by experimenters) of demands are said to fall within the same domain.

Two examples follow for clarification. The first analysis is of simple addition and begins at General Model of Hierarchical Complexity order 3—nominal actions (Dromi, 1984). Table 1 shows the breakdown of addition and distribution. Saying a number constitutes the simplest task, represented as s_n. The sentential stage action (General Model of Hierarchical Complexity stage 2a) of saying numbers in sequence, constitutes an operation on two or more nominal s_n operations, ordering them in a nonarbitrary way. This is written as $S(s_i \ldots s_j)$. That is, the sentential action S acts on saying the numbers i to j. Continuing this concatenation according to Table 2, the primary stage action of addition is $A(C, C, C)$, or the primary act of adding (A) operating on three ordered instances of counting (C).

The second example is taken from the wash problem (Commons, Miller, & Kuhn, 1982), a task where subjects are asked to predict the outcome of washing a dirty cloth in one or more variable conditions (water temperature, type of soap, etc.) based on previous outcomes for other cloths. The response at the primary stage is like a law ''a cloth washed in hot water comes out clean.'' Participants act as if they use the syllogism and make a correct prediction. Table 3 gives the action breakdown. Note that the preoperational step is not the consequence of the action, but merely the time ordering of the events.

TABLE 3
Task Analysis of the Wash Problem: Stage of Causality

General model of hierarchical complexity order	Action	Representation
3. Sensory-motor	Consistently taking the ‘‘clean cloth’’ rather than the ‘‘dirty cloth’’ no matter what it is made of	a_c
4. Nominal	Saying ‘‘clean’’	s_c
5. Sentential	Saying ‘‘I washed the cloth in hot water.’’	$S(s_I \ldots s_w)$
	Ordering clean and dirty	O_{cd}
6. Preoperational	Storytelling: ‘‘I washed the cloth in hot water and then it came out clean’’	$T(S_w, S_c)$
7. Primary	Logical step (syllogism): wash in hot water → clean ‘‘I washed in hot water, thus the cloth is clean’’	$L(O_{cd}, T(S_w, S_c), S_c)$
8. Concrete	Episodic causality: ‘‘If I washed the cloth in hot water with liquid soap, blue booster, and Brand A bleach, then the cloth came out clean’’	
9. Abstract	Cloths can be: clean or dirty (different state of the cloth)	y_1
	Washed: in hot or cold water (different temperature water)	x_1
	With liquid or powdered soap (different kinds of soap)	x_2
	With blue or pink booster (different kinds of booster)	x_3
	With Brand A or B bleach (different kinds of bleach)	x_4
10. Formal	Univariate Empirical and Logical Causality	
	The water temperature, x_1, makes the cloth come out clean or dirty, y_1	$(x_1)(x_1 \rightarrow y_1)$
11. Systematic	Multivariate Empirical and Logical Causality (arrays of causes)	
	It is the water temperature, x_1, or the brand of bleach, x_4, that makes the cloth come out clean or dirty, y_1	$S_1 = (x_1, x_4)(x_1 \text{ or } x_4 \rightarrow y_1)$
	It is the water temperature, x_1, and the brand of bleach, x_4, that make the cloth come out clean or dirty, y_1	$S_2 = (x_1, x_4)(x_1 \text{ and } x_4 \rightarrow y_1)$
12. Metasystematic	Multisystem Empirical and Logical Causality (comparing arrays of causes)	$T[S_1] = S_2$
	Seeing that the following systems may be transformed into the other by replacing or with and. Noting there are many more or cases than and cases	$T[(x_1, x_4)(x_1 \text{ or } x_4 \rightarrow y_1)] = (x_1, x_4)(x_1 \text{ and } x_4 \rightarrow y_1)$
	It is the water temperature, x_1, or the brand of bleach, x_4, that makes the cloth come out clean or dirty, y_1	
	It is the water temperature, x_1, and the brand bleach, x_4, that make the cloth come out clean or dirty, y_1	

Note. The letters in the representations are somewhat arbitrary. The letter *a* stands for actions; *s*, sort saying; *S* for saying in order; *O* is ordering outcomes; *T* is telling a story; *L* is for logically operating on elements of story. The letter *x* stands for an input or stimulus; the letter *y* for output or outcome; the letter *S* stands systems with multiple inputs or stimuli.

IDEAL TASK PERFORMANCE

Researchers can construct tasks specifically to detect the stage of performance of a subject in a given domain (Feldman, 1980). The only known way to measure stage of performance is to observe a subject's response to specific tasks and then compare the actual performance to the possible ways that the task could be accomplished. The method presented here rejects traditional notions of ''correct'' or ''rational'' task performance, preferring instead the notion of the *ideal task performance.* Ideal task performance is the successful completion of parameters or definitions of the problem. It differs in each case according the task specified. For example, in basketball, if the ball goes through the net, then it satisfies the ideal task performance. Ideal performance conforms to the behavior of some ideal performer who maximizes performance on any given task. That is, the ideal performer produces the most nearly complete and efficient performance on a task possible. But there is no necessary connection between ideal performance and actual performance in contrast to Chomsky's doctrine of competence (e.g., Chomsky, 1957). A formal description of task requirements does not mean that participants use any part of the formal description to find correct solutions.

A particular participant's order of performance is measured by the highest order task that that subject performs adequately (i.e., relative to the ideal task performance). We prefer to use the Rasch (1980) scaled measure as discussed further on. When researchers measure actual subject performance on tasks designed to measure stage in a particular domain, the scaled performance is an empirical measure of stage. However, it is important to make the distinction between the order of the hierarchical complexity of the ideal task performance and the answer to the problem given by a task. If the task in question is the addition problem $1 + 1$, with symbolically written numbers, then the answer to the problem is 2. However, the outcome of the ideal performance at the level of primary operations is the formula $1 + 1 = 2$. This is the evidence for the gapping between stages. There is no way to get a paradigm of the ordering from within this task. The problem is equivalent to proving the validity of the metaformula $1 + 1 = 2$ when the problem only allows the use of numbers as solutions.

DEFINITION OF ORDER OF PERFORMANCE

Order of performance on one task is not necessarily generalizable to other tasks, even where the tasks share the same order of complexity and are found in the same domain. Other task properties such as horizontal complexity can account for that. Yet performance on tasks is the only basis for measuring order. This problem is solved if researchers use more than one series of tasks. In general, the hierarchical complexity of the task refers to the complexity of the relationships that must be discriminated for ideal task performance. Order of performance is thus an indication of the most complex discrimination the subject makes on a given task under the conditions of the stimulus situation. No claims are made as to cognitive structures of the brain or about some overall stage of competence in the subject. Inferences can be drawn about the ability of the subject to discriminate stageable signals based on performance of tasks, however. By the definition of task complexity, there can be no *décalage* in the hierarchical ranking of tasks. Tasks are rigidly defined by their analytic breakdown, and domain remains an ambiguous classification superimposed on the rigorous task structure by experimenters. Within subject performance, however, *décalage* is expected. Hence, this method maintains the rigorous definition of a ‘‘hard’’ stage model, but with the added subtlety of locating where these criteria fall.

SUMMARY OF AXIOMS AND THEOREMS

Order Axioms

Based on the preceding definitions, it is now possible to begin to define a set of formal standards that must be satisfied to establish a consistent concept of stage. Here we will briefly describe the axioms; a more extensive description may be found in the Appendix. The notion of *entity* serves as a point of departure. An entity is a set (or equivalence class) of tasks having the same order of hierarchical complexity. Entities must satisfy these three requirements for forming a *sequence:*

Axiom 1: Entities are nontrivial. Every entity must contain at least one potentially detectable task (i.e., for any entity X, there exists some task x).

Axiom 2: Entities are connected. There is no logical indeterminancy. The order of any entity is equal to, greater than, or less than the order of any other entity, but not more than one of these relations holds for any two entities.

Axiom 3: Entities are transitive. If the order of any entity A is greater than the order of some entity B, and the order of B is greater than some entity C, then the order of A is greater than the order of C.

Entity Sequence Axioms

Axiom 4: Inclusivity. Entity n contains entity n-1 inclusively. Inclusivity means that the higher order action can do all that the lower order actions can do and more.

Axiom 5: Discreteness. The immediate successor of the entity of order n is the entity of order $n + 1$. The entities are discontinuous.

Relative Complexity of Actions Axioms

Axiom 6: Formation of actions from prerequisites. For one task-required action to be higher in the chain than a second action, the second action must be a prerequisite for the first action.

Axiom 7: Relational composition. A task-required action must organize two or more distinct, earlier actions in the chain. (R. M. Dunn, personal communication, January 26, 1986.)

Axiom 8: Order of definition. The order of the organizing action and what it acts upon in the chain is fixed.

THEOREMS RESULTING FROM THE AXIOMS

A system of orders of hierarchically tasks exists in any case in which all of the above axioms are satisfied. A stage of performance system parallels such a system. The following theorems are proofs derived from these axioms and are demonstrated only informally.

Existence of Orders of Hierarchical Complexity and Resulting Stages

Theorem 1: Orders of hierarchical complexity exist. That collections of actions can be sequenced into orders of hierarchical complexity rests upon Axiom 6, which defines what is meant by qualitative difference. This discreteness or ‘‘gap’’ axiom requires that there be no interpolated action between sets of new required acts and the sets of previous order acts. For example, someone has performed an action (distribution) required by a distribution task at the concrete order.

COROLLARY 1: STAGES EXIST. If orders of hierarchical complexity of tasks exist, then there are actions that perform those tasks.

The discovery of one case in which the gap axiom and the other order of hierarchical complexity axioms are satisfied is sufficient to logically demonstrate the existence of stages and stage sequences.

Theorem 2: Postformal hierarchical tasks and performance exist. As hierarchical complexity increases, the nature of the gap between each order of complexity changes. The gap from the primary to the concrete order involves only the coordination of addition and multiplication to form distributivity. In later-order gaps, such as the one from the systematic to the metasystematic order, at the metasystematic order, one has to create an entire metalanguage and set of metarules in order to coordinate the operations of a previous systematic order (Commons & Richards, 1984a; The systematic order is also called the consolidated formal operational stage; Kohlberg, 1990; Pascual-Leone, 1984).

Making a deduction within a formal operational system requires formal operations. Showing that something is true about a formal-operational system requires systematic operations. Showing that something is true about a systematic formal-operational system requires metasystematic operations.

Theorem 3: A linear order may exist only within a single domain, on single sequences of tasks. Axioms 4 through 7 are not so restrictive as to allow for this lattice structure, but are restrictive enough to require linear sequences within a single task sequence.

This result can be stated as follows. When one sequence of task performances in time is projected onto another sequence of task performances, the combined sequences do not necessarily form a linear order. The task sequences may have to be from the same domain, and the same subdomain.

Theorem 4. There is only one sequence of orders of complexity in all domains. The order numbers describe the same complexity of task-required actions irrespective of domain. Thus one can map any developmental sequence onto any other. This result does not imply synchronous development. Whereas the stage numbers may be the same, the stages of performance may develop at different times.

From an analytic perspective, the task requirements are constant and unvarying for different individuals regardless of how the subject feels about the task. The order complexity of each task within a sequence of tasks can be directly compared to the order of complexity for another set of tasks. The non-order of complexity aspects of tasks only makes it more difficult to apply Axioms 8 through 10.

Theorem 5. Hierarchical complexity is a linear, ordinal scale. The scale of order of hierarchical complexity maps into the positive and negative integers and the admissible transformations are of the form $mx + n$, where m is a positive integer and n is any integer. So if x is an integer, so is the transformed $mx + n$. Hierarchical complexity forms a linear ordinal scale, which is a new type of measure. It has the following properties.

$$O_i \text{ are ordinal numbers, } O_i \in \boldsymbol{O}.$$

They are linear:

$$O_j = mO_i + b,$$

where m and b are ordinal numbers, $m \in \boldsymbol{O}$, $b \in \boldsymbol{O}$.

PROOF: By Axiom x, each action y coordinates at least two lower stage actions. The orders of hierarchical complexity increase in number of actions by at least twofold for every recursion. The order, O, is greater than or equal to 2^o, the O^{th} power of 2. Such powers have the linear property $y = mx + b$. In this case $y = O_j$, $x = O_i$. Hence the orders of hierarchical complexity have the linear property.

We have made some scaled measurements of stage of items and stage of respondent answers (Dawson, Goodheart, Draney, Wilson, & Commons, in press). We used Rasch Analysis (1980), which jointly minimizes errors for both items and respondents. What was interesting about the results is they showed the items of a given order of hierarchical complexity all clustered around the corresponding stage number and were roughly equally spaced.

Theorem 6. Measures of performance. Whereas the gaps between orders of the complexity of tasks are discrete, measurement is continuous. Each discrete performance on a given stage task (actual or inferred) either succeeds (1) or fails (0).

CONCLUSION

Removing any axiom from the above model leaves orders of hierarchal complexity (and therefore stage of performance) undefined. Adding more axioms would either reduce the generality of hierarchical complexity unnecessarily or make the axioms inconsistent. No claim is made as to the uniqueness of the axiom system. The General Model of Hierarchical Complexity shows that because a single measuring system represents hierarchical complexity, only one stage sequence underlies all domains of development. To have more than one sequence, another measure of hierarchical complexity would have to exist.[1] All tasks have some complexity associated with them. Thus, all tasks have stages associated with them. Because orders of complexity require such large jumps in performance (Fischer et al., 1984) even though development may be continuous (Acredolo, 1995; Brainerd, 1978), it may appear as jumps or gaps (Commons & Calnek, 1984) on stage measures (Dawson, Goodheart, Draney, Wilson, & Commons, in press; Rosales & Baer, 1997).

Establishing an analytic measure of stage has many benefits for psychology. First, by classifying task complexity analytically, such a model produces measures that are independent of observation and of actual subject performances. This leads to a greater degree of accuracy and consistency in stage measurement. Second, because the model defines a single sequence underlying all domains of development, it sets forth the core requirements of stages in every domain (see Kohlberg & Armon, 1984). Although many stage researchers posit more core requirements for stage, none require fewer. The set given by this model may allow for a systematic presentation of the consensus of theorists as to these core requirements. Indeed, the presence of a definitional set of axioms even makes it possible to determine whether a particular developmental theory also qualifies as a stage theory. For, according to this model, any theory that fails to account for the hierarchical complexity of task in the definition of development stage will by definition fail to yield results that are accurate, or even significant and meaningful as to order of developmental complexity.

Some doubt may remain as to whether there exists only one stage sequence. For example, if there were more than one way to perform a task, would this lead to alternative orders (and hence disagreement) as to the proper stage of a task and the true stage sequence? The answer, to a certain extent, is ''yes.'' There will inevitably be some argument over the validity of certain task analyses. The fact that the analysis can be done by no means implies that it will be obvious or easy in all cases. It is possible to define our stage sequence such that it is generated from the task analysis with the shortest possible task chains, however. This will eliminate some ambiguity. Additionally, it may be asked whether it is possible to know from a single task tree that another tree will not differ, such that complexity two on a one-task tree falls between complexity two and complexity three on another.

[1] This does not imply the ''structured whole'' notion of Piaget.

The model, however, defines tasks that have two concatenations as tasks of complexity two, regardless of how difficult they may be to perform. ‘‘Falling between complexities’’ is therefore not a possibility. The General Model of Hierarchical Complexity, then, shows that, because a single measuring system represents hierarchical complexity, only one stage sequence underlies all domains of development. For more than one sequence, another measure of hierarchical complexity would have to exist, although this by no means implies the structure-of-the-whole notion of Piaget. Such an alternative measure has not been identified, however. Moreover, because every task is of a certain order of hierarchical complexity, all tasks have a stage of performance associated with the response that they require for optimal resolution. Stage of performance on any given task will correspond to the order of hierarchical complexity of the task itself.

This model therefore answers some of the most fundamental questions that are asked about stage theories. By theoretically presenting a method for the analysis of tasks, and deriving an actual task chain, the model demonstrates that such chains exist. It also shows that stage sequence is invariable across all domains, because domain has been removed from the construction of the task sequence and so has no implications for task complexity. Consequently, task complexity remains unchanged regardless of how broadly or narrowly domains are defined. Finally, the model offers an analytic model of stage development, based upon a set of mathematically grounded axioms. The axiomatic nature of the model entails that stages exist as more than *ad hoc* descriptions of sequential changes in human behavior and formalizes key notions implicit in most stage theories. As such it offers clarity and consistency to the field of stage theory and to the study of human development in general. It also lays the basis for a new form of computational complexity compatible with neural networks.

APPENDIX

Order Axioms

Based on the preceding definitions, it is now possible to begin to define a set of formal standards that must be satisfied to establish a consistent concept of hierarchal complexity and resulting stage of performance. The notion of *entity* serves as a point of departure. An entity is a set (or equivalence class) of tasks having the same order of hierarchical complexity. Entities must satisfy these three requirements for forming a *sequence:*

Axiom 1: Entities are nontrivial. Every entity contains at least one task (i.e., for any entity X, there exists some task x). This axiom insures that order of hierarchical complexities of tasks and the corresponding stages of performance will refer to actions and elements in some real domain. That is, these tasks must be potentially detectable. For example, in the laundry causality problem, formal operations are used to correlate the relation between causal variables of water temperature, bleach brand, booster color, and soap form and outcome variables of clean and dirty; the set of variables is not empty.

Axiom 2: Entities are connected. There is no logical indeterminacy. The order of any entity is equal to, greater than, or less than the order of any other entity, but not more than one of these relations holds for any two entities. That is, $C(A) > C(B)$, $C(A) < C(B)$, or $C(A) = C(B)$. $C(X)$ is the complexity of X. This axiom is necessary to show that orders of hierarchical complexity and the resulting stages will be comparable and that every hierarchically ordered task will belong to some order. For example, it can be determined in a causality problem whether the successful correlation of several variables is at a higher or lower order of complexity than the action of successfully isolating the variables themselves or whether it is at an order of complexity equivalent to that isolation action. It follows from the definition of concatenation, then, that relations between variables are always higher, that is, more complex, than the single variables that they contain as elements.

Axiom 3: Entities are transitive. If the order of any entity A is greater than the order of some entity B, and the order of B is greater than some entity C, then the order of A is greater than the order of C. That is, $(x)[C(A) > C(B)$ and $C(B) > C(C)]$ then $(x)[C(A) > C(C)]$. $C(X)$ is the complexity of X. For example, isolating systematic causal relations is at higher order of complexity than isolating simple causality, which is itself at a higher order than identifying a single variable. Thus isolation of systematic causal relations necessarily demonstrates a higher complexity than the identification of a single variable.

This numerical sequence of entities now takes the form of a hierarchy that satisfies some of our stage criteria. *Sequences of orders of hierarchical task complexity* are sets of entities that satisfy the sequence Axioms 4 and 5, in addition to the sequence Axioms 1 through 3. Axiom 4 actually follows directly from the definitions of concatenation and task structures.

Sequence Axioms

Axiom 4: Inclusivity. Entity n contains entity $n - 1$ inclusively. If some demands of entity n do not belong to entity $n - 1$, but all demands of entity n belong to entity $n + 1$, then the demands can be arranged in an inclusive order. Inclusivity means that higher-order of complexity actions will be more powerful than lower order actions, in the sense that the higher order of complexity action can do all that the lower order of complexity actions can do and more. For example, the discovery of causal relations in the laundry problem is more powerful than the simple awareness of the hot/cold water variable.

Axiom 5: Discreteness. The immediate successor of the entity of order n is the entity of order $n + 1$. This criterion makes the sequence of entities discontinuous and stage-like. In the laundry problem, for example, between statements about the variables themselves and statements about the relation between variables, no intermediate statement can be made.

Relative Complexity of Actions Axioms

To fill the requirements for a theory of hierarchical complexity and therefore a stage theory, the following three axioms must be satisfied. These axioms define the meaning of one task being of hierarchical complexity than another task.

Axiom 6: Formation of actions from prerequisites. For one task-required action to be higher in the chain than a second action, the second action must be a prerequisite for the first action. This axiom can also be illustrated by the laundry problem in which the recognition of variables not only precedes the recognition of causality but is also necessary to it. That is, the recognition of variables is a prerequisite for the recognition of causality.

Axiom 7: Relational composition. A task-required action must organize two or more distinct, earlier actions in the chain (R. M. Dunn, personal communication, January 26, 1986). This organization stands in contradistinction to a simple chain of actions at the same level, where there are no prerequisites and hence no organization of predecessors.

Axiom 8: Order of definition. The order of the organizing action and what it acts upon in the chain is fixed. That is, the position of the organizing action in question in the chain relative to the prerequisite actions cannot be rearranged in principle. Thus in the laundry problem, the order of the outcome (clean) and the causal variable (hot water) cannot be reversed. In mathematics, this axiom can be illustrated with the principle of distributivity, in which the functions must be performed in a given order. This axiom insures that the order of the chain of hierarchical complexity of actions is not an arbitrarily defined sequence of events, as is the case in some stage theories. There must not be any other possible organization at a fundamental level without changing the definition of the final action.

Limit on the Number of Possible Orders and Stages

Any given task has a discrete number of measurable steps and a small number of observable elements. The relational composition axiom requires that each organizing action coordinate at least two of the $n - 1$ components of an action, where n is the order of the organizing action. For example, there are two actions organized by an order-two task, and an order-three task organizes two of those. Order-three actions organize four lower order actions. Now, suppose there were 36 orders and corresponding stages. Then an order-36 task would organize 2^{35} (two to the thirty-fifth) sensory-motor actions, or over 34 billion actions. If each takes 1 s, then this task would require nearly 10 million h to complete. Because most stimulus situations contain more restrictive time constraints, and because human life itself is finite, the relational composition axiom entails the existence of an upper limit on the number of possible stages.

Axioms That Have Not Been Included

(a) Sudden or abrupt emergence. Competencies occur all of a sudden, in an all-or-none stage change. Reason for rejection: No empirical way to decide at this time (Commons & Calnek, 1984; but see Fischer, Hand, & Russell, 1984; and Rosales & Baer, 1997, for a contrary view).

(b) Operations within stages have certain logicomathematical properties. These are the operations such as identity, negation, reciprocation, correlative transformations, and the mathematical relations of complementarity, commutativity, associativity that are associated with certain groups). Reason for rejection of the Inhelder and Piaget (1958) INRC group: Task analysis and performance analysis show that these are not the operations demanded or used by subjects at any order (Brainerd 1978; Commons, Richards & Kuhn, 1982; Ennis, 1978). The same holds for lower stage logicomathematical properties.

(c) Closure. There is no completeness at any stage of any set of operations or actions or assumptions other than those found in Euclidean Geometry and Simple Logic (see Campbell & Bickhard, 1986, for critique of the significance Piaget places on closed structures such as the INRC group).

(d) There is an ultimate order and corresponding stage. Reason for rejection: No evidence can support this. Someone may discover a next stage (Habermas, Wednesday, October, 15, 1986, personal communication; Commons & Richards, 1984b). Kohlberg and Fischer have indicated that there is an ultimate stage, although Piaget (see Chapman, 1988) has not agreed.

Psychological Assumptions That Were Not Included

(a) Ensemble d'ensemble (cross-domain structure). For Piaget, the doctrine of *structures d'ensemble* (overarching structures) applies within domains, not across them. See Chapman (1988). The strong evidence for this was that development takes place at the same rate in all domains. Reason for rejection: A good deal of research (e.g., Kohlberg, 1984) finds horizontal

décalage when non-stage task difficulty is not controlled. Weinreb and Brainerd (1975) find nonsynchronous development to be pervasive.

(b) Earlier schemes drop out. Reason for rejection: Organisms use the lowest stage that works satisfactorily because that requires least effort.

(c) There is no regression. Reason for rejection: Research showed (Denton & Krebs, 1990; Krebs, Denton, Vermeulen, Carpendale, & Bush, 1991) that alcohol drinking events as well as other events reduce stage.

(d) Operatory stages are universal, performance at the highest stages occurs everywhere. Reason for rejection: Piaget (1972) accepted the evidence and dropped the claim that formal operations were universal. Dasen (1977) indicated that formal operations tested with Inhelder's and Piaget's (1958) tasks are not found in nonschooled cultures.

THEOREMS RESULTING FROM THE AXIOMS

A system of orders of hierarchical tasks exists in any case in which all of the above axioms are satisfied. A stage of performance system parallels such a system. The following theorems are proofs derived from these axioms and are demonstrated only informally.

Existence of Orders of Hierarchical Complexity and Resulting Stages

Theorem 1: Orders of hierarchical complexity exist. Any developmental sequence of tasks can easily satisfy the first five axioms. The crux of the argument is that collections of tasks can be sequenced into orders of hierarchical complexity. The results require that stage actions may also be ordered. This argument rests upon Axiom 6, which defines what is meant by qualitative difference. This discreteness or ''gap'' axiom requires that there be no interpolated action between sets of new required order of acts and the sets of previous order of acts. Brainerd (1978), among others, argues that Axiom 6 is false, because every action can be divided into a series of smaller actions and that no ''gap'' between types of actions exists. Likewise, he argues that every performance can be broken down into smaller performances.

The discovery of one case in which the gap axiom and the other order axioms are satisfied is sufficient to logically demonstrate the existence of stages and stage sequences. Failure to find one case would reject this existence theorem. The distributivity case is presented here as a very clear-cut example. This example is generalized to the case with *groups* in the Appendix. The case of distributivity shows how all of the axioms are necessary to explain the relationship between two tasks at differing levels.

DISTRIBUTIVITY. Distributive tasks require actions outside of the boundaries of simple addition and multiplication. In the task sequence generated by the General Model of Hierarchical Complexity within the arithmetic domain, distributive tasks ($f_\times$) on numbers require *concrete*[2] order actions ($f_\times$ belongs to Entity C), while addition (e_{i+}) and multiplication ($e_{j\times}$) tasks on numbers require *primary*-order actions (e_{i+}, $e_{j\times}$ belong to Entity P) (see Table 1 and Commons & Richards, 1984a). We will show that Entity C is higher than and separate from Entity P and therefore forms part of an ordered task sequence. The gap between the complexity of operations needed to perform addition and multiplication and the complexity of distributivity thus demonstrates a gap in task order and resulting stage. In order to illustrate this theorem, a numeric example is presented first, followed by a counterexample that does not meet the conditions of the theorem.

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[2] The *concrete* operational stage requires the full coordination of primary operations on numbers. We will show that distributivity on actions on numbers is such a coordination.

NUMERIC EXAMPLE. Addition, action e_{i+}, and multiplication, action $e_{j\times}$, are actions belonging to Entity P that coordinate rational numbers not including 0 for the inverse of multiplication (Elements). Distributivity, action a_3, is an action that coordinates additive and multiplicative acts. For example, let the set of elements be the natural numbers. Addition, action e_{i+}, of 5 and 7 and of 10 + 14 is expressed as

$$(5 + 7) = 12, (10 + 14) = 24 \tag{1}$$

Multiplication, action $e_{j\times}$, of 2 and 12, 2 and 5, and 2 and 7 is expressed as

$$(2 * 12) = 24, (2 * 5) = 10, (2 * 7) = 14. \tag{2}$$

Numerical distributivity is shown as

$$2 * (5 + 7) \text{ equals } (2 * 5) + (2 * 7). \tag{3}$$

The 2 that multiplies the sum is distributed across each element of the sum (action e_j). What was an operation in the previous order becomes an element for the new order.[3]

Below, it will be shown that the distributive act coordinates these additive and multiplicative acts. The following expression represents each instance of multiplication or addition in a trinary relation. In more abstract terms, the trinary relations in equations 2 and 3 can be written as follows:

$$\text{action } e_j(\text{element } a_1, \text{ element } a_2, \text{ outcome } a_3) = e_1(a_1, a_2, a_3) \rightarrow a_1\ o\ a_2 = a_3, \tag{4}$$

In this expression, o stands for one instance of either of the two classes of actions, addition or multiplication. The subscripted a's stand for the elements (numbers) to which one of these actions is applied. In this expression it is critical to distinguish between a_3 and the entire expression which is Entity *C:*

$$a_1\ o\ a_2 = a_3. \tag{5}$$

There is a strong tendency to think of a_3 as the stage product, rather than simply as the product of the equation. While a_3 is the outcome of applying addition to the elements, 1 and 2, and is therefore the third term of the additive relation, it is not the stage product. The additive relation 1 + 2 = 3 is the stage product. This product is the coordination of preoperational stage products, namely, sequenced numbers. A *stage product* is an entire scheme of coordination from beginning to end, which includes the actions and elements from that particular stage of coordination. It is this product that becomes an element for action at the next stage. The fact of the stage product shows that Axioms 8 and 9 are satisfied and thus that a gap is formed between stages when a new stage product is created. In the following example, by contrast, a concatenation of actions that does not produce a new stage action is presented.

COUNTEREXAMPLE: ADDITION AS AN ITERATIVE ACTION. Instances of the trinary relation that occur in an addition and multiplication table respectively are as follows:

$$e_{1+}(1, 2, 3) \rightarrow 1 + 2 = 3.$$

$$e_{1\times}(1, 2, 2) \rightarrow 1 \times 2 = 2.$$

These are but two instances of a large number of additive and multiplicative primary-stage actions, or e_{ij}'s, each of which is a trinary relation.

Start with the set of elements $E = \{a_i\} = \{1, 2, 3, 5, 6, 9\}$ and two actions, addition and multiplication, e_+ and $e_\times$. Let the actions be applied to any pair of the elements {1, 2, 3, 6}.

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[3] The distributivity example can also use logic instead of integer arithmetic: Logically, the statement, ''*A* is true and (*B* is false or *C* is true)'' is equivalent to the statement ''(*A* is true and *B* is false) or (*A* is true and *C* is true).''

The application of stage actions to stage elements $e_{ij}(a_i)$ yields the products

$$
\begin{aligned}
1.\ e_{+}(a_i) \rightarrow\ & e_{1+}(1, 2, 3) \rightarrow\ 1 + 2 = 3 \\
& e_{2+}(1, 3, 4) \rightarrow\ 1 + 3 = 4 \\
& e_{3+}(2, 3, 5) \rightarrow\ 2 + 3 = 5 \\
& e_{4+}(3, 6, 9) \rightarrow\ 3 + 6 = 9 \\
2.\ e_{\times}(a_i) \rightarrow\ & e_{1\times}(1, 2, 2) \rightarrow\ 1 \times 2 = 2 \\
& e_{2\times}(1, 3, 3) \rightarrow\ 1 \times 3 = 3 \\
& e_{3\times}(2, 3, 6) \rightarrow\ 2 \times 3 = 6 \\
& e_{4\times}(3, 3, 9) \rightarrow\ 3 \times 3 = 9
\end{aligned}
$$

This is a subset of possible stage products for the simple application of actions to elements. Other stage products are possible if actions are applied iteratively to elements. For example, if addition is composed twice, more stage products become possible:

$$
\begin{aligned}
3.\ e_{+}(e_{+}(a_i)) \rightarrow e_{2+}(1,(e_{1+}(1, 2, 3)), 4) \rightarrow\ & 1 + (1 + 2) \\
& = 1 + (3) \\
& = 4 \\
\rightarrow e_{3+}(2,(e_{1+}\ (1, 2, 3)), 5)\ \text{etc.} &
\end{aligned}
$$

This is an example of nonhierarchical composition because any e_{+} could be directly applied to the numbers. The order of application does not matter. There is no coordination of operations or relations, other than the repetition of the e_{+}. These compositions are not at the concrete-operational stage because all of the actions within the composition are at the primary-operational stage and could be directly applied to the numbers.

Organization of Routines Is Not Arbitrary

For a composition of actions to be at a higher order than the actions being composed, an action must directly apply only to the actions of the previous order and not directly to elements of those actions. An example of a hierarchical composition of the actions defined above (e_{1+} through $e_{4\times}$) would be an action, $f_{1\times}$, that organizes multiplication and addition distributively. Distribution poses the problem of how to interpret multiplication on addition of numbers. The subject does not know whether to begin with addition or multiplication. If addition is done first, then the problem is to know what to multiply. If multiplication is done first, the problem is to know what number(s) needs to be multiplied. This problem can be shown with the example of multiplying the sum of 1 and 2. While it may seem obvious what the correct procedure is, it is not obvious to the beginning concrete-operational thinker. The concatenation of two actions, multiplying and adding,

$$f_{1\times}(e_{\times},\ e_{+}(a_i),\ e_{+}(e_{\times}(a_j)))$$

must be worked out in terms of actual cases. In the following, the more general expression of distribution is given and then converted into the specific terms in which it would be worked out at the concrete-operational stage;

$$
\begin{aligned}
4.\ & f_{1\times}(e_{\times},\ e_{+}(a_i),\ e_{+}(e_{\times}(a_j))) \rightarrow \\
& f_{1\times}(3,(e_1 + (1, 2, 3,)),\ [e_{4p}\ (e_{2\times}(1, 3, 3),\ e_{3\times}\ (2, 3, 6), 9)])
\end{aligned}
$$

This complex expression is just a form of a trinary relation, $f_{\times}(a, b, c)$, where $f_{\times}$ is multiplication (on sums). This relation can be thought of as the conventional expression $a \times b = c$. Each letter is equal to

$$f_{1\times}(a, b, c) \rightarrow a \times b = c.$$

This is the operation of multiplication over the three elements

$a = 3$. The number 3 is a primary-operational hierarchical complexity element.

$b =$ the trinary relation $e_{1+}(1, 2, 3) \rightarrow 1 + 2 = 3$.

$c =$ the complex trinary relation, $[e_{4+}(e_{2\times}(1, 3, 3), e_{3\times}(2, 3, 6), 9)]$,

which has the form $e_{4+}(r, s, t) \rightarrow r + s = t$, where

$$r = e_{2\times}(1, 3, 3) \rightarrow 1 \times 3 = 3$$

$$s = e_{3\times}(2, 3, 6) \rightarrow 2 \times 3 = 6$$

$$t = 9.$$

Thus, $r + s = t$, or 9, is the sum of 1 times 3 and 2 times 3. Expression 4 may have r, s, and t substituted into it:

$$\text{4a. } f_{1\times}(3, e_1 + (1, 2, 3,), [e_{4+}(e_{2\times}(1, 3, 3), e_{3\times}(2, 3, 6), 9)]) \rightarrow$$

$$\text{4b. } f_{1\times}(a, b, c) = f_{1\times}(a, b, e_{4+}(r, s, t)) \rightarrow$$

$$3 \times (1 + 2) = (1 \times 3) + (2 \times 3) = 9.$$

The most critical aspect of this discussion has been the demonstration that very Entity C is situated at the successor stage of Entity P. According to Axiom 7, if $f_{1\times}$ is at a higher level than $e_{+}(a_i)$ and $e_{\times}(a_i)$, then this succession will be true. Acting on and organizing elements and products from the previous stage will meet our definition of how hierarchical composition generates an act at a successor stage.

First, from the expression $f_{1\times}(a, b, c)$, it was shown that $f_{1\times,}$ which belongs to Entity C, was not directly acting upon the elements of the actions of Entity P alone. Of the three elements organized by the trinary relation $f_{1\times}$, only $a = 3$ is an a_i element of e_{+} from the Entity P. The relation $f_{1\times}$ acts on element b, which is a previous stage product defined by $e_{1+}(1, 2, 3)$. The $f_{1\times}$ act cannot organize the predecessor stage elements 1 and 2 directly. The same is true for element c, which is the product of the organization of two previous stage products, $c = e_{4+}(e_{2\times}(a_i), e_{3\times}(a_j), a_k)$. If these products were not defined, distribution would not be defined. Axiom 8 is thus satisfied.

Second, because the $f_{1\times}$ organizes the actions and their elements from Entity P, Axiom 9, the relational composition axiom, is satisfied. The $f_{1\times}$ acts upon prerequisite stage products and coordinates them. More specifically, $f_{1\times}$ acts on e_{1+}, e_{4+}, $e_{2\times}$ and $e_{3\times}$.

Third, the distribution act organizes specific acts of addition and multiplication into a given order. It establishes an order relation across these acts and elements, as is specifically shown in 4a. Actions must be performed from the innermost parentheses to the outermost. Because the order defined by the coordination is invariant, Axiom 10 is satisfied.

The above example demonstrates how the operations at one order of hierarchical complexity action become an outcome that is operated on by the new order hierarchical task's operations. This transformation of an action into an element is what Piaget meant by operations on operations.[4] Because the operations at the previous stage are necessary to form elements that are

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[4] See Stein and Commons (1987, May) and Commons and Hallinan (1990) for the psychology of reflection.

operated on at a new stage, the order of the chain of elements and outcomes cannot be reversed. This is why a stage sequence must be made up of discrete sets of actions and why the sequence cannot be rearranged. (In other words, it cannot be imagined otherwise.) Through the distributivity property example we can see that there exists *hierarchical composition of actions,* which results in a chain of entities forming a stage sequence. Thus the case of distributivity illustrates the actual existence of a stage sequence.

Theorem 2: Postformal orders and resulting stages exist. As stages increase, the nature of the gap between them changes. The gap from the primary to the concrete order of complexity involves only the coordination of addition and multiplication to form distributivity. In the later order of complexity gaps, such as the one from the systematic to the metasystematic order, one has to create an entire metalanguage and set of metarules in order to coordinate the operations of a previous systematic order (Commons & Richards, 1984a; Kohlberg, 1990; and Pascual-Leon, 1984, called this consolidate a formal-operational stage).

There is an unfillable gap between showing that things are true within an arithmetic system and showing that things are true about such a system. This gap is demonstrated by Gödel's (1930/1931/1977) incompleteness theorem. In the proof that arithmetic is incomplete, Gödel's theorem also shows that axioms 8, 9, and 10 are satisfied. The axioms *of* arithmetic are not sufficient to demonstrate all the truths *about* arithmetic. There is thus at least one truth that can be found only with metalogic. A corollary of Gödel's incompleteness theorem shows that consistency is a property of a system not provable by methods formalized from within that system (Luchins & Luchins, 1965). There are metamathematical propositions that are not decidable within the axiom system. ''The prefix *meta* has been used to identify a theory (or a language) in which the object of study is itself a theory (or a language)'' (p. 185). Completeness is defined in terms of the axioms from within a system. Consistency must necessarily be proved from outside that formal axiom system.

Axiom 8 states that for an action to be higher in the chain than another action, the first action must be a prerequisite for the second. To prove consistency of a system, one must first have the system defined in the chain. In Gödel's proof, the systematic stage actions are represented by axioms within an axiom system. The metasystematic actions are the proof of consistency of the axiom system. For an axiom system to be proven consistent or complete, Gödel's proof shows that the proof must be made from the next higher stage.

Axioms 9 and 10 are also satisfied by Gödel's proof. The proof of consistency for an axiom system necessarily makes the axiom system an element in a larger coordination of systems (axiom 9). A consequence of this coordination is that the chain of organization of axioms, axiom systems, and metamathematical systems is fixed and cannot be rearranged in principle (axiom 10).

Making a deduction within a formal operational system requires formal operations. Showing that something is true about a formal-operational system requires systematic operations. Showing that something is true about a systematic formal-operational system requires metasystematic operations.

Order, Sequence, and Measure across Tasks and Domains

We propose that whereas hierarchical complexity is a linear structure, stage is a lattice structure. A lattice structure is weaker than a linear structure in that it does not require cross-domain synchrony. In a linear order, if $a > b$, $b > c$, then $a > c$. In a lattice, however, one may have $a > b$, $b > c$, and a and c not comparable.

Theorem 3: A linear order of development may exist only within a single domain, on single sequences of tasks. A linear order across domains and tasks is the algebraic, topological statement of Piaget's (1953) *structures d'ensemble* (structures of the whole). Domain refers to the area of which a task is a member. Domain of a task can be conceptualized horizontally, while stage can be conceptualized vertically and hierarchically. Domain may be viewed as hierarchically organized in a reductionistic way, e.g., biology is based on chemistry which in turn is based on physics. Nonsynchronous development is described by *décalage* in horizontal organization. Feldman (1980) suggests in detail how to understand the organization of a domain. If the structure-of-the-whole holds, performance measured in two different domains should show the same stage during the same time period (synchrony). Axioms 4 through 7 define a lattice structure. These axioms are not so restrictive as to force linear structure, but are restrictive enough to require linear sequences within a single task sequence. Both Case's (1985) and Fischer's (1980) models used to require synchrony, whereas their models as well as Campbell & Bickhard's (1986) model require only a lattice structure.

The relationship between sets of entities, and not sets of structures of the whole, is what gives rise to stages. We have already shown existence of stage within a domain and task.

Theorem 4: There is only one sequence of hierarchical complexity of tasks in all domains.

DOMAIN AND STAGE. Historically, stage theories utilized the notion that tasks at different stages within the same domain are ''qualitatively different'' (Kohlberg & Armon, 1984). Stage of a task within a domain is determined by applying the last three axioms, 8 through 10. To determine the order of hierarchical complexity of a task one must count the number of actions in the hierarchy leading up to the task-required action. Each action in the hierarchy organizes actions from the previous stage. Axioms 8 through 10 describe the conditions under which an action will be vertically higher in stage than actions from the previous stage.

By checking repeatedly to see if an action requires a previous stage action which in turn requires a previous stage action, one can determine the linear hierarchy for a single sequence of tasks. The stage numbers describe the same complexity of task-required actions irrespective of domain. Thus one can map any developmental sequence onto any other. This result does not imply synchronous development.

Stage systems must propose a parallel set of stages or levels referred to here as circular, sensory, nominal, preöperational, primary, concrete, abstract, formal, systematic, metasystematic, cross-paradigmatic, and so on (see Table 2).[5] This parallelism is exemplified by the work of Fischer (1980), and Campbell and Bickhard (1986), and by the work of others including Armon's (1984) ethical development, Kohlberg's (1984) moral development, and Loevinger's ego development (Loevinger & Blasi, 1976; Loevinger & Wessler, 1970) as systematized by Cook-Greuter (1990) and Pascual-Leone's (1984) organismic theory of life stages.

COROLLARY 1. COMMON MEASURES OF STAGE OF COMPLEXITY ACROSS DOMAINS EXIST: REQUIREMENTS NEEDED TO ACCESS STAGE. The task requirements necessary to solve a problem may have both hierarchical and stage and nonhierarchical properties. Let the hierarchical properties be a in Table 3. The nonstage properties b_1, b_2, and b_3 only make it more difficult for researchers to assess the stage of a task in a sequence. For example, asking people to add six numbers, b_2, in their head rather than two numbers, b_1, increases nonstage demands. A subject

[5] This table shows that it is possible to show precise correspondence between sequences from different theories (adapted from Alexander, Druker, & Langer, 1990; Commons, Richards, & Armon, 1984; Sonnert & Commons, 1994). The following sequences have been modified by extending the stages and levels upward and filling in the nominal stage (Dromie, 1984). Fischer does not posit any levels beyond 10 (personal communication, March 28, 1988). Kohlberg (e.g., Colby & Kohlberg, 1987a,b; Kohlberg, 1984, 1987) repeatedly reproduced a table from Colby and Kohlberg's unpublished 1975 paper. After the appearance of cognitive developmental stages beyond formal operations, Kohlberg (1990) modified this correspondence a number of times (Commons, Richards, & Armon, 1984, final table; Kohlberg, 1981, 1990; Schrader, 1987). Although he (personal communication, June 20, 1985) revised his section of the final table for Commons, Richards, and Armon (1984), he never reviewed the last changes on his own table (Kohlberg & Ryncarz, 1990). For a discussion of this issue, see Commons and Grotzer (1990), Walker (1986), and Sonnert and Commons (1994).

may also report that a lower stage task in an unfamiliar domain is harder than a higher stage task in a familiar domain.

From Theorem 4, we know that the order of complexity of a task always exists for sequences of tasks. When explanations of performance are part of the task performed, the minimal set of actions may include remembering the actions, as well as naming and reflecting upon the actions (King, Kitchener, Wood, & Davison, 1989; Kitchener & King, 1990; Tappan, 1990). Again, additional nonstage requirements make a task more difficult than a task without them. The additional requirements do not raise the order of the task but might raise the age of a given subject at which a task is successfully completed. For example, requiring an action that concatenates a set of actions in an ordered way has stage properties while requiring a certain amount of material to be recalled does not directly have stage properties (see Axiom 4).

From an analytic perspective, the task requirements are constant and unvarying for different individuals regardless of how the subject feels about the task. The complexity of stage for each task within a sequence of tasks can be directly compared to the complexity of stage for another set of tasks. The nonstage aspects of tasks only make it more difficult to apply axioms 8 through 10.

Concatenating Complexity

Theorem 5: Hierarchical complexity is a linear ordinal scale. The scale of order of hierarchical complexity maps into the positive and negative integers. The admissible transformations are of the form $mx + n$, where m is a positive integer and n is any integer. So if x is an integer, so is the transformed $mx + n$. Hierarchical complexity forms a linear ordinal scale. This scale is the same type of number system that was used before fractions and negative numbers were introduced into arithmetic. It has the properties

$$O_i \text{ are ordinal numbers, } O_i \in O$$

They are linear:

$$O_j = mO_i + b, \text{ where } m \text{ and } b \text{ are ordinal numbers, } m \in O, b \in O$$

PROOF: By Axiom x, each action y coordinates at least two lower stage actions. The orders of hierarchical complexity increase in number of actions by at least twofold for every recursion. The order, O, is greater or equal to 2^O, the O^{th} power of 2. Such powers have the linear property $y = mx + b$. In this case $y = O_j$, $x = O_i$. Hence the orders of hierarchical complexity have the linear property. We have created the Laundry and Balance Beam Series covering primary through systematic order of complexity. We used a Rasch analysis (1980), which jointly minimizes errors for both items and respondents. The results showed the items of a given order of hierarchical complexity all clustered around the corresponding stage number and were roughly equally spaced.

COROLLARY: VARYING THE OBJECT'S COMPLEXITY SHOULD HAVE THE SAME EFFECT AS VARYING THE ACTION'S COMPLEXITY

How researchers and subjects represent actions, events, and situations in a task changes with stage. By definition, stage increases as the complexity of the actions and discriminations of relations increases. Our conception of stage is based on a stage-generating mechanism. The number of times the mechanism can be applied is infinite, although researchers only understand approximately 14 such applications. Task complexity is defined by the General Model of Hierarchical Complexity metric discussed below (Commons & Richards, 1984a). Measuring complexity allows one to differentiate the stage of a task from the difficulty of a task. Task difficulty may be correlated with the age at which subjects complete tasks and with difficulty of IQ test items.

Piaget recognized that what children reasoning at different stages considered rational differed from what adults considered rational. What constitutes rationality changes with stage.

Piaget and Kohlberg posited that formal operational stage action reflected rational reasoning. But Piaget also stated that making formal operational systems required reflections upon them. Rational thought to Kohlberg is logical, consistent, reversible—having all the formal operational properties. The proof of the existence of postformal stages (Commons, Richards & Kuhn, 1982) demonstrated that formal operations were not sound reasoning in many cases and hence not rational. Because the sequence of stages is infinite, none of the postformal forms of rationality are entirely adequate either; each new stage's rationality resolves problems with the former stage's rationality. Thus how researchers represent a task and the task's solution will always be limited by stage of the researcher.

Measurement of Performance

Commons and Richards (1984a, 1984b), Fischer, Hand, and Russell (1984), Klahr and Wallace, (1976), Case (1982), Pascual-Leone (1984), and Siegler (1986) measure stage of performance through task analysis. Stage of performance has been measured after researchers make a thorough task analysis of a series of tasks (Commons & Richards, 1984a,b; Campbell & Bickhard, 1986). Task analysis is useful in isolating the steps of a task that are prerequisites for later steps of the task.

Each task used to assess stage can be solved by a minimal set of required actions. The minimal set of actions may include nonverbal actions, verbal actions, or both. There are no tasks that have only one solution; the number of nonminimal solutions is infinite.[6] Because there are a number of possible solutions to any task, task analysis as well as an analysis of actual performances yields more solutions to a task than those specified by the problem's creators. Scoring performance by using task analysis takes the fact of all possible solutions into consideration.

The ''psychologic'' of stage consists of the implicit decision rules with which a subject acts. These rules describe what a subject does in relationship to the task to be performed and the conditions present surrounding the task. Order of performance in a domain is usually characterized by the highest stage tasks that are performed adequately and consistently. How optimal the measurement conditions are will influence the order at which a subject performs (Commons, Grotzer, & Davidson, in preparation). As Campbell and Bickhard (1986) state, there is no way to factor out the conditions and obtain an underlying competence as implied by Fischer's (Fischer, Hand, & Russell, 1984) interpretations that substantial physiological changes accompany spurts in performance. A task may be solved at an order higher than the nominal stage of the task. While one infers the possible ways the tasks can be accomplished, there is no necessary simple correspondence between the nominal and the actual stage.

Theorem 6: Measures of performance: While the gaps between stages of a task are discrete, measurement is continuous. Each discrete performance on a given stage task (actual or inferred) either succeeds (1) or fails (0). A common set of performance measures exists if conditions of choice theory are met in problem construction and interviewing. Performance consists of a mixture of lower and higher stage acts. Probability of acts at a given stage thus goes from 0 to 1, as shown in probability theory, when there is sampling over more than one instance (see Commons & Calnek, 1984). We have proposed that choice theory and signal detection (Commons & Rodriguez, 1993; Ellis, Commons, Rodriguez, Grotzer, & Stein, 1987, June; Kantrowitz, Buhlman, & Commons, 1985, April; Munsinger, 1970) be used to ascertain stage of performance from performance data. More recently, we recommend using Rasch (1980; Spada & McGraw, 1985) and Saltus analyses (Spada & McGraw, 1985; Wilson, 1989). We do find gaps in some performance, depending on how well the sequence of questions or scoring was done.

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[6] Any solution can be made longer by applying an operation and its inverse or, for that matter, the identity operation.

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